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AN ENERGETIC APPROACH TO HOMOGENIZATION PROBLEMS WITH RAPIDLY OSCILLATING POTENTIALS

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ABSTRACT

~~We show~~ It is shown that many, a priori distinct, problems of homogenization including the case of rapidly oscillating potentials ~~(cf. Bensoussan, J. L. Lions and Papanicolau [2])~~, can be studied, and the limit problem computed, in a unified way, through general compactness and convergence results for sequences of functionals of calculus of variations. The convergence notion is taken in variational sense, more precisely ~~we use~~ by the notion of Γ-convergence introduced by De-Giorgi. [4].

Tau

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SIGNIFICANCE AND EXPLANATION

In Mechanics, Physics, Chemistry it occurs frequently one has to study a boundary value problem in media with periodic structure. When the period of the structure is small compared to the size of the domain in which the system is studied, a microscopic description of the system is difficult, but one might expect to get a good macroscopic description of the system by making the period-parameter go to zero in the equations which describe it. This type of process is called "homogenization".

When the equations have a variational formulation; for example, when the solution u_ε of the microscopic problem (with period ε) minimizes a functional F_ε (which is in general related to the "energy" of the system) over a space X (the boundary conditions are included in X), one looks for a limit functional F_0 such that u, the limit of u_ε, minimizes F_0 over X. We can say that F_0 is the limit of the sequence F_ε in the "variational sense". In recent years this type of convergence, called "Γ-convergence" (notion introduced by E. De Giorgi) has been intensively studied.

Using recent results of compactness and convergence (in Γ-sense) for a large class of functionals of calculus of variations we can attack many different problems of homogenization with a unified point of view. For example, one can explain the behaviour of u_ε, solution of the following equation with "rapidly oscillating potentials" (studied by Benssoussan, Lions, and Papanicolau) $\mu u_\varepsilon - \Delta u_\varepsilon + \frac{1}{\varepsilon} W(\frac{x}{\varepsilon}) u_\varepsilon = f$ on Ω; $u_\varepsilon|_{\partial\Omega} = 0$, as ε goes to zero. W is a periodic function (in each variable) from $\mathbb{R}^n$ to $\mathbb{R}$, with zero mean value. Actually, one can prove that u_ε converges and compute the limit equation.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

AN ENERGETIC APPROACH TO HOMOGENIZATION PROBLEMS WITH RAPIDLY OSCILLATING POTENTIALS*

Hedy Attouch†

Introduction. The model problem, studied by B.L.P. [2], of homogenization with rapidly oscillating potentials, is the following:

Let W be an Y-periodic function (Y is a basic cell in $\mathbb{R}^n$) with zero mean value, and u_ε the solution (which exists for μ large enough) of:

$$(I_\varepsilon) \qquad \mu u_\varepsilon - \Delta u_\varepsilon + \frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}\right) u_\varepsilon = f \text{ on } \Omega, \ u_\varepsilon\big|_{\partial\Omega} = 0 .$$

When ε goes to zero, one can prove that u_ε converge weakly in $H_0^1(\Omega)$ to u solution of:

$$(I) \qquad \mu u - \Delta u + M(WX)u = f \text{ on } \Omega; \ u\big|_{\partial\Omega} = 0 .$$

We denote by $M(WX)$ the mean value of WX and X is defined by:

$$\left| \begin{array}{l} \Delta X = W \\ X \text{ is } Y\text{-periodic} . \end{array} \right.$$

a) Since u_ε converge in weak-$H_0^1(\Omega)$ and $\frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}\right)$ converge (to zero) in weak-$H^{-1}(\Omega)$ it is rather surprising one can go to the limit on the product $\frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}\right)u_\varepsilon$! Moreover its limit depends on the partial differential operator you work with in the equation (I_ε). (Here we took the Laplacian). We shall first give a direct energetic solution to the model problem, and emphasize on the fact that the sequence $\left(\frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}\right)u_\varepsilon\right)_{\varepsilon>0}$ converge to $M(WX)u$ in weak-$H^{-1}(\Omega)$ and that one cannot expect a stronger convergence.

b) Then, we give an energetic interpretation of the previous problem showing that it is a particular case of the more general problem which consists in computing the

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limit (in variational sense) of sequences of functionals of the following type:

$$F_\varepsilon(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, u, Du\right)dx$$

where f is Y-periodic in x, and convex in (u,Du), and satisifes boundedness and coerciveness assumptions.

This approach leads us to study more general problems than (I_ε) and to some conjectures concerning the convergence of such general sequences F_ε:

c) We obtain, through direct computational method, the limit equation when f is quadratic in (u,Du); so we treat in a unified way, homogenization problems with first order terms (cf. [2]), with oscillating potentials, ---. As an example, let us consider

$$(II_\varepsilon) \qquad \mu u_\varepsilon - \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_j}\right) + \frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}\right)u_\varepsilon = f \quad \text{on} \quad \Omega; \quad u_\varepsilon|_{\partial\Omega} = 0$$

where the coefficients a_{ij} are in $L^\infty(\mathbb{R}^n)$, Y-periodic, and the associated second order operators uniformly elliptic. Don't assume the a_{ij} symmetric. We observe that (μ is taken large enough) the sequence $\left(\frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}\right)u_\varepsilon\right)_{\varepsilon>0}$ converge to a first order term in weak-$H^{-1}(\Omega)$ and prove that u_ε converge weakly in $H_0^1(\Omega)$ to u solution of:

$$(II) \qquad \mu u + \mathcal{A}(u) + \sum_i M\left(\alpha_i - \sum_j \alpha_j \frac{\partial X^i}{\partial x_j}\right)\frac{\partial u}{\partial x_i} + M(WX)u = f \quad \text{on} \quad \Omega; \quad u|_{\partial\Omega} = 0$$

where $\mathcal{A}$ is the classical homogenized operator of the family $A^\varepsilon = -\sum \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right)$, (cf. Theorem 1 for a complete statement and definitions of α_i, X^i, X). When the coefficients are symmetric, $a_{ij} = a_{ji}$, (II) reduces to

$$(II)_{bis} \qquad \mu u + \mathcal{A}(u) + M(WX)u = f \quad \text{on} \quad \Omega; \quad u|_{\partial\Omega} = 0 .$$

d) We then study natural extensions of the previous results for higher order operators, studying with particular attention the limit in variational sense of the functionals

$$F_\varepsilon(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, u, Du, D^2u\right)dx ,$$

when f is Y-periodic in x, and quadratic in (u, Du, D^2u), (cf. Theorems 3, 4, 5). This allows us to describe, for example, the limit of u_ε solution of

$$(\mathrm{IV})_\varepsilon \qquad \mu u_\varepsilon + \Delta^2 u_\varepsilon + \frac{1}{\varepsilon^2} W_\varepsilon u_\varepsilon = f$$

and more generally of u_ε solution of

$$(\mathrm{VIII})_\varepsilon \qquad \mu u_\varepsilon + \Delta^2 u_\varepsilon + W_{0,\varepsilon} \Delta u_\varepsilon + \frac{1}{\varepsilon} W_{1,\varepsilon} \operatorname{div} u_\varepsilon + \frac{1}{\varepsilon^2} W_{2,\varepsilon} u_\varepsilon = f$$

as ε goes to zero; (we assume the W_i have zero mean value) (cf. Théorem 6).

Then we study the general compactness and convergence problem (Theorem 7 and Theorem 8).

Plan.

I. Homogenization with rapidly oscillating potentials for second order operators

1.1. Study of the model problem.

Proposition 1. Let W be a Y-periodic function with zero mean value; as ε goes to zero, the solution u (which exist, if taking μ large enough) of

$$(I_\varepsilon) \qquad \mu u_\varepsilon - \Delta u_\varepsilon + \frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}\right) u_\varepsilon = f \quad \text{on } \Omega; \quad u_\varepsilon\big|_{\partial\Omega} = 0$$

converge in weak-$H_0^1(\Omega)$ to u solution of

$$(I) \qquad \mu u - \Delta u + M(WX)u = f \quad \text{on } \Omega; \quad u\big|_{\partial\Omega} = 0$$

where X is defined by (1.1) $\begin{cases} \Delta X = W \\ X \text{ is Y-periodic} . \end{cases}$

Proof of Proposition 1.

a) The existence of u_ε solution of (I_ε), for μ large enough, follows easily from the coerciveness of the bilinear form $a_\varepsilon(\cdot,\cdot)$ associated with (I_ε): (from now on, given G a function on $\mathbb{R}^n$, we shall write $G_\varepsilon(x) = G(\frac{x}{\varepsilon})$). From (1.1) we get

(1.2) $\Delta(\varepsilon X_\varepsilon) = \frac{1}{\varepsilon} W_\varepsilon$; it follows that

$$\int_\Omega \frac{1}{\varepsilon} W_\varepsilon u^2 dx = \int_\Omega \Delta(\varepsilon X_\varepsilon) u^2 dx = 2 \int_\Omega (DX)_\varepsilon \cdot u \cdot Du dx ,$$

and

$$a_\varepsilon(u,u) \geq \mu |u|_2^2 + |Du|_2^2 - 2|DX|_\infty |u|_2 \, |Du|_2 \geq \rho_0 \|u\|^2_{H_0^1}$$

for μ large enough and some $\rho_0 > 0$. From the uniform coerciveness of the $(a_\varepsilon)_{\varepsilon>0}$, the $(u_\varepsilon)_{\varepsilon>0}$ are bounded in $H_0^1(\Omega)$.

b) Let $u_\varepsilon \xrightarrow{w - H_0^1} u$, as ε goes to zero and let us identify u as the solution of (I): The only problem is to compute the limit in weak-$H^{-1}(\Omega)$ of the sequence $\left(\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon\right)_{\varepsilon>0}$: Given $\varphi \in C_0^\infty(\Omega)$ (a C^∞ function with compact support) let us look to

$I_\varepsilon = \int_\Omega \frac{1}{\varepsilon} W_\varepsilon u_\varepsilon \varphi dx$; from (1.2)

$I_\varepsilon = \int_\Omega \Delta(\varepsilon X_\varepsilon) u_\varepsilon \varphi dx$; integrating by parts

$I_\varepsilon = \int_\Omega \varepsilon X_\varepsilon [\Delta u_\varepsilon \varphi + 2Du_\varepsilon D\varphi + u_\varepsilon \Delta\varphi] dx$ and using that u_ε satisfied (I_ε)

$I_\varepsilon = \int_\Omega cX_\varepsilon[(\mu u_\varepsilon + \frac{1}{\varepsilon} W_\varepsilon u_\varepsilon - f)\varphi + 2Du_\varepsilon D\varphi + u_\varepsilon \Delta\varphi]dx$; when ε goes to zero

$\lim_{\varepsilon\to 0} I_\varepsilon = \lim \int_\Omega cX_\varepsilon \frac{1}{\varepsilon} W_\varepsilon u_\varepsilon \varphi dx = M(WX) \int_\Omega u\varphi dx$ that is to say

$\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon \xrightarrow[\varepsilon \to 0]{w - H^{-1}} M(WX)u$ and u satisfies the limit equation (I).

<u>Remark 1</u>. From (1.2) we get:

$$\forall v \in H_0^1(\Omega), \int_\Omega \frac{1}{\varepsilon} W_\varepsilon u_\varepsilon v dx = \int_\Omega \Delta(cX_\varepsilon)u_\varepsilon v dx = \int_\Omega (DX_\varepsilon)[Du_\varepsilon v + u_\varepsilon Dv]dx$$

$$\leq 2|DX|_\infty |u_\varepsilon|_{H_0^1} |v|_{H_0^1} \leq C|v|_{H_0^1} .$$

In fact we cannot expect a better estimate, than the $H^{-1}(\Omega)$ one, on the sequence $(\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon)$: If the $(\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon)_{\varepsilon>0}$ were bounded in L^2, then $\mu u_\varepsilon - \Delta u_\varepsilon$ would be bounded in H^2, the u_ε would be bounded in H^2, hence compact in H_0^1; since $\frac{1}{\varepsilon} W_\varepsilon$ goes to zero in $w - H^{-1}$, the product $\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon$ would go to zero, which is not the case if $W \neq 0$.

<u>Remark 2</u>. A rather unsuspected result is that, dispite the fact that the sequence $(\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon)_{\varepsilon>0}$ is bounded only in H^{-1}, its limit is still a zero order term; actually this is relevant to the particular form of the equation (I_ε); we shall see next a more general situation where this is no more the case (cf. Theorem 1).

<u>Remark 3</u>. The limit of $\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon$, which is equal to $(-\frac{1}{|Y|}\int_Y |DX|^2 dy)u$, depends on the differential operator which governs the equation: since X is defined by $\Delta X = W$, if instead of (I_ε) we consider (for simplicity)

$$\mu u_\varepsilon - \lambda\Delta u_\varepsilon + \frac{1}{\varepsilon} W_\varepsilon u_\varepsilon = f \quad (\lambda > 0) ,$$

then, the limit equation will be:

$$\mu u - \lambda\Delta u + \frac{1}{\lambda^2} M(WX)u = f .$$

1.2. <u>Energetic interpretation</u>.

Let us interpret now the behaviour of u_ε solution of

$$(I_\varepsilon) \qquad \mu u_\varepsilon - \Delta u_\varepsilon + \frac{1}{\varepsilon} W_\varepsilon u_\varepsilon = f \text{ on } \Omega, \quad u_\varepsilon|_{\partial\Omega} = 0 .$$

a) The solution u_ε of (I_ε) minimizes the functional $F_\varepsilon(\cdot) - \langle f,\cdot\rangle$ where

$$F_\varepsilon(u) = \int_\Omega \frac{1}{2}\mu|u|^2 + \frac{1}{2}|Du|^2 + \frac{1}{2\varepsilon} W_\varepsilon |u|^2 dx ;$$

In fact, introducing X the solution of (1.1) $\begin{cases} \Delta X = W \\ X \text{ Y-periodic} \end{cases}$, which exists since $\int_Y W(y)dy = 0$, we may write F_ε in the following way:

$$F_\varepsilon(u) = \int_\Omega \frac{1}{2}\mu|u|^2 + \frac{1}{2}|Du|^2 + \frac{1}{2}\varepsilon\Delta X_\varepsilon \cdot u^2 dx \quad \text{and integrating by parts ,}$$

$$= \int_\Omega \frac{1}{2}\mu u^2 + \frac{1}{2}|Du|^2 - (DX)_\varepsilon u\cdot Du\, dx .$$

Therefore

$$F_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, u, Du)dx$$

with

$$f(x,\zeta,z) = \frac{1}{2}|z|^2 + \sum_i \frac{\partial X}{\partial x_i}(x)\zeta z_i + \frac{1}{2}\mu\zeta^2 .$$

Taking μ large enough in order f to be positive (for example $\mu > \frac{1}{2}\sum |\frac{\partial X}{\partial x_i}|^2$ a.e.), f will be a positive quadratic form in (ζ,z) and hence convex; moreover $\forall(x,\zeta,z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, $\lambda_0|z|^2 \leq f(x,\zeta,z) \leq \Lambda_0|(\zeta,z)|^2$. It follows that the corresponding functional F_ε is strictly convex coercive on $H_0^1(\Omega)$ and that the only solution of the corresponding Euler equation (I_ε) is the unique point u_ε, where F_ε assume its minimum.

b) From H. Attouch [1], (Theorem 2.1) it follows that the family of convex functionals $(F_\varepsilon)_{\varepsilon>0}$ is compact with respect to the $\Gamma^-(w - H_0^1)$ and all $\Gamma^-(L^p)$ convergence. (This result is an extension of the compactness result of Carbone and Sbordone [3] to the case where the functionals depend on u and Du, which is precisely our situation.)

Let us give the precise statement we use:

Theorem A (cf. [1]). Let $f_h : (x,\zeta,z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \longmapsto f_h(x,\zeta,z)$ a sequence of functions measurable in x, convex continuous in (ζ,z) positives, $f_h(x,0,0) = 0$; let us define for every Ω bounded open set in $\mathbb{R}^n$

$$F_h(u,\Omega) = \begin{cases} \int_\Omega f_h(x,u(x),Du(x))dx & \text{if } u \in \mathrm{Lip}_{loc}(\mathbb{R}^n) \\ +\infty & \text{if } u \in L^1_{loc}(\mathbb{R}^n) \setminus \mathrm{Lip}_{loc}(\mathbb{R}^n) \end{cases}$$

and let us assume:

(H) $\exists a \in L^1_{loc}(\mathbb{R}^n)$ and φ continuous increasing in $|\xi|, |z|$ such that

$$\forall h \in \mathbb{N}, \qquad 0 \le f_h(x,\xi,z) \le a(x)\varphi(|\xi|,|z|) .$$

Then, the following conclusion holds:

α) There exist a subsequence $(h(p))_{p \in \mathbb{N}}$ such that $\forall \Omega$ bounded open set in $\mathbb{R}^n$, $\forall u \in \mathrm{Lip}_{loc}(\mathbb{R}^n)$

$$F(u,\Omega) = \Gamma^-(L^1(\Omega)) \lim_{\substack{p\to+\infty \\ v\to u}} F_{h(p)}(v,\Omega) = \Gamma^-(L^\infty(\Omega)) \lim_{\substack{p\to+\infty \\ v\to u}} F_{h(p)}(v,\Omega) \quad \text{exist.}$$

β) There exists a function f measurable in x, convex continuous in (ξ,z) such that $\forall \Omega$ bounded open set, $\forall u \in \mathrm{Lip}_{loc}(\mathbb{R}^n)$

$$F(u,\Omega) = \int_\Omega f(x,u(x),Du(x))dx .$$

γ) If $\lambda_0|z|^p \le f_h(x,\xi,z) \le \Lambda_0(1 + |(\xi,z)|^p)$ for some $p > 1$ then the conclusion extends to the whole space $W^{1,p}_0(\Omega)$ and $\forall \Omega$ bounded open set, $\forall u \in W^{1,p}_0(\Omega)$

$$F(u,\Omega) = \Gamma^-(w - W^{1,p}_p(\Omega)) \lim_{\substack{p\to+\infty \\ v\to u}} F_{h(p)}(v,\Omega) .$$

So we may consider the problem of homogenization (I_ε) as a particular case (more precisely of quadratic type) of the general problem of computing the limit in variational sense of the following sequence of functionals

(P) $$F_\varepsilon(u) = \int_\Omega f(\tfrac{x}{\varepsilon}, u, Du)dx \quad \text{where } f \text{ is } \begin{cases} Y\text{-periodic in } x \\ \text{convex in } (\xi,z) \end{cases}$$

and satisfies some boundedness and coerciveness assumptions; for simplicity let us assume

$$\lambda_0|z|^p \le f(x,\xi,z) \le \Lambda_0(1 + |\xi|^p + |z|^p) \quad \text{with } p > 1 .$$

What the preceding theorem tells us is that, if F^ε converge, its limit F_0 is still of the form $F_0(u) = \int_\Omega f_0(x,u,Du)dx$: this implies that the sequence $(u_\varepsilon)_{\varepsilon>0}$, where u_ε minimizes F_ε converge to u which minimizes F_0.

Two particular cases of problem (P) have been intensely studied:

1. $f(x,\xi,z) = f(x,\xi)$ (f is independent of z); from Marcellini and Sbordone [?] it follows that F_ε converge to F_0 where

$$F_0(u) = \int_\Omega f_0(u)dx \quad \text{and} \quad f_0(\xi) = \frac{1}{|Y|}\int_Y f(x,\xi)dx \; .$$

2. $f(x,\xi,z) = f(x,z)$ (f is independent of ξ); it follows from Carbone and Sbordone [3] that F_ε converge to F_0 where

$$F_0(u) = \int_\Omega f_0(Du)dx \quad \text{and} \quad f_0(z) = \min\left\{\frac{1}{|Y|}\int_Y f(x,Du+z)dx/u \text{ Y-periodic}\right\} .$$

We are now going to study in the next paragraph the general situation corresponding to the model problem, that is to say the case where f is quadratic in (ξ,z). (We may also notice that the case $f(x,\xi,z) = f_1(x,\xi) + f_2(x,z)$, is a straightforward extension of the previous ones.)

1.3. <u>Homogenization results for quadratic integral functionals in (u,Du).</u>

The general form of a quadratic integral functional F_ε will be:

$$F_\varepsilon(u) = \int_\Omega \left\{ \sum_{i,j} a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum b_i\left(\frac{x}{\varepsilon}\right) u \frac{\partial u}{\partial x_i} + c\left(\frac{x}{\varepsilon}\right)u^2(x)\right\}dx$$

where the coefficients a_{ij}, b_i, c are Y-periodic; let us assume that

$$\sum a_{ij}(x)z_i z_j \geq \lambda|z|^2 \qquad (a_{ij} = a_{ji})$$

and

$$c(x) \geq \mu$$

with μ large enough in order the F_ε to be convex, positive, uniformly coercive on $H_0^1(\Omega)$.

The Euler equation, $\nabla F_\varepsilon(u_\varepsilon) = f$, ($f \in H^{-1}(\Omega)$) can be written:

$$(1.3) \qquad -\sum \frac{\partial}{\partial x_i}\left(a_{ij_\varepsilon} \cdot \frac{\partial u_\varepsilon}{\partial x_j}\right) - \sum \frac{\partial}{\partial x_i}(b_{i_\varepsilon} u_\varepsilon) + \sum b_{i_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i} + 2c_\varepsilon u_\varepsilon = f$$

or equivalently

$$(1.4) \qquad -\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u_\varepsilon}{\partial x_j}\right) - \frac{1}{\varepsilon}\left[\sum\left(\frac{\partial b_i}{\partial x_i}\right)_\varepsilon\right]u_\varepsilon + 2c_\varepsilon u_\varepsilon = f \; .$$

More generally, let us consider

$$(II_\varepsilon) \qquad -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u_\varepsilon}{\partial x_j}\right) + \frac{1}{\varepsilon} W_\varepsilon u_\varepsilon + c_\varepsilon u_\varepsilon = f \quad \text{on } \Omega; \quad u_\varepsilon|_{\partial\Omega} = 0$$

with a_{ij} non-necessary symmetric, W Y-periodic with zero mean value; without loss of generality we may assume $c \equiv \mu$; the following theorem gives us the answer to the asymptotic comportment of u_ε as $\varepsilon \to 0$:

Theorem 1. For μ large enough, the solution u_ε of

$$(II_\varepsilon) \qquad \mu u_\varepsilon - \sum \frac{\partial}{\partial x_i}\left(a_{ij_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_j}\right) + \frac{1}{\varepsilon} W_\varepsilon u_\varepsilon = f; \quad u_\varepsilon|_{\partial\Omega} = 0$$

exists, and u_ε converge (as ε goes to zero) to u solution of:

$$(II) \qquad \mu u + \mathcal{A}(u) + \sum_i M\left(\alpha_i - \sum_j \alpha_j \frac{\partial X^i}{\partial x_j}\right)\frac{\partial u}{\partial x_i} + M(WX)u = f \quad \text{on } \Omega; \quad u|_{\partial\Omega} = 0$$

(1.5). $\mathcal{A}$ is the homogenized operator of the family $(A^\varepsilon)_{\varepsilon>0}$, $A^\varepsilon u = -\sum \frac{\partial}{\partial x_i}\left(a_{ij_\varepsilon}\frac{\partial u}{\partial x_j}\right)$:

$$\mathcal{A} = -\sum_{i,j} q_{ij}\frac{\partial^2}{\partial x_i \partial x_j} \qquad q_{ij} = M\left(a_{ij} - \sum_k a_{kj}\frac{\partial X^i}{\partial x_k}\right) \quad \text{and}$$

X^i is defined by $\left|\begin{array}{l} A^*(X^i - x_i) = 0 \ \text{ with } \ A^* = -\sum_{i,j}\frac{\partial}{\partial x_i}\left(a^*_{ij}\frac{\partial}{\partial x_j}\right) \\ X^i \text{ is Y-periodic} \quad (a^*_{ij} = a_{ji}) \end{array}\right.$

(1.6). X is defined by $\left|\begin{array}{l} A^*X + W = 0 \ \text{ and,} \\ X \text{ is Y-periodic} \end{array}\right.$

$$(1.7) \qquad \alpha_j = \sum_k a^*_{jk}\frac{\partial X}{\partial x_k} .$$

When the A^ε are symmetric $(a_{ij} = a_{ji})$ or with constant coefficients the limit equation (II) reduces to

$$(II)_{bis} \qquad \mu u + \mathcal{A}(u) + M(WX)u = f .$$

Corollary 1. When ε goes to zero the sequence of convex functionals $(a_{ij} = a_{ji})$

$$F_\varepsilon(u) = \int_\Omega \sum_{i,j} a_{ij_\varepsilon}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum b_{i_\varepsilon} u \frac{\partial u}{\partial x_i} + \frac{1}{2} c_\varepsilon u^2 dx$$

converge in $\Gamma^-(w - H_0^1)$ sense (we assume the a_{ij} uniformly elliptic and $c \geq \mu$ with μ large enough) to F_0 which is equal to

$$F_0(u) = \int_\Omega \left\{ \sum_{i,j} q_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{1}{2} M(c + WX)u^2 \right\} dx$$

where q_{ij} are defined in (1.5) (with $A^* = A$) and X is defined by $\left| \begin{array}{l} AX = \sum_i \frac{\partial b_i}{\partial x_i} \\ X \text{ is Y-periodic .} \end{array} \right.$

<u>Proof of Theorem 1.</u>

a) As in the model problem the existence of u_ε, solution of (II_ε) follows from the coerciveness of the bilinear form $a(\cdot,\cdot)$ associated with (II_ε): From (1.6) one gets

$$(1.6)_{bis} \qquad A_\varepsilon^*(\varepsilon X_\varepsilon) + \frac{1}{\varepsilon} W_\varepsilon = 0$$

and

$$\begin{aligned} \int_\Omega \frac{1}{\varepsilon} W_\varepsilon u^2 dx &= \int_\Omega \sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij}^* \frac{\partial(\varepsilon X_\varepsilon)}{\partial x_j} \right) u^2 dx \\ &= -2 \int_\Omega u \sum_{i,j} a_{i,j_\varepsilon}^* \left(\frac{\partial X}{\partial x_j} \right)_\varepsilon \frac{\partial u}{\partial x_i} dx \\ &\leq 2 \sup_{i,j} |a_{ij}|_\infty \cdot |DX|_\infty \cdot |u|_2 \cdot |Du|_2 \,. \end{aligned}$$

It follows that

$$a_\varepsilon(u,u) \geq \mu |u|_2^2 + |Du|_2^2 - C|u|_2 \cdot |Du|_2 \geq \rho_0 \|u\|_{H_0^1}^2$$

for μ large enough and some $\rho_0 > 0$. From the uniformly coerciveness of the $(a_\varepsilon(\cdot,\cdot))_{\varepsilon>0}$ it follows that the $(u_\varepsilon)_{\varepsilon>0}$ are bounded in $H_0^1(\Omega)$. Let $u_\varepsilon \xrightarrow{w - H_0^1} u$; it is clear that $(A^\varepsilon u_\varepsilon)_{\varepsilon>0}$ and $(\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon)_{\varepsilon>0}$ are bounded in $H^{-1}(\Omega)$; we are going to compute their limits in weak-H^{-1}:

b) First, look to the limit of $(\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon)_{\varepsilon>0}$ as ε goes to zero; from $(1.6)_{bis}$,

$$\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon = -A^{\varepsilon *}(\varepsilon X_\varepsilon) u_\varepsilon = u_\varepsilon \sum \frac{\partial}{\partial x_i} \left(a_{ij}^* \frac{\partial X}{\partial x_j} \right)_\varepsilon ;$$

let us introduce

(1.7) $$\alpha_i = \sum a^*_{ij} \frac{\partial \chi}{\partial x_j} ;$$

then

$$\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon = u_\varepsilon \sum_i \frac{\partial}{\partial x_i} (\alpha_{i_\varepsilon})$$

which is equal in distribution sense to

(1.8) $$\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon = \sum_i \frac{\partial}{\partial x_i} (\alpha_{i_\varepsilon} u_\varepsilon) - \sum_i \alpha_{i_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} .$$

The first term of the second member clearly converge in weak-H^{-1};

(1.9) $$\sum_i \frac{\partial}{\partial x_i} (\alpha_{i_\varepsilon} u_\varepsilon) \xrightarrow[(\varepsilon \to 0)]{w - H^{-1}} \sum_i M(\alpha_i) \frac{\partial u}{\partial x_i} .$$

The second one is bounded in $L^2(\Omega)$, but we cannot compute directly its limit; given $\varphi \in C_0^\infty(\Omega)$ let us consider

$$J_\varepsilon = - \int_\Omega \sum_i \alpha_{i_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} \varphi dx = - \int_\Omega \sum_i \left(\sum_j a^*_{ij_\varepsilon} \frac{\partial}{\partial x_j} (\varepsilon X_\varepsilon)\right) \frac{\partial u_\varepsilon}{\partial x_i} \varphi dx$$

$$J_\varepsilon = - \int_\Omega \sum_j \frac{\partial (\varepsilon X_\varepsilon)}{\partial x_j} \cdot \left(\sum_i a^*_{ij_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\right) \varphi dx \quad \text{and integrate by parts}$$

$$= \int_\Omega \varepsilon X_\varepsilon \left[\varphi \sum_{i,j} \frac{\partial}{\partial x_j} \left(a^*_{ij_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\right) + \sum_{i,j} a^*_{ij_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\right] dx$$

$$J_\varepsilon = \int_\Omega \varepsilon X_\varepsilon \left[-\varphi \cdot A^\varepsilon u_\varepsilon + \sum_{i,j} a^*_{ij_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\right] dx .$$

We now use that u_ε is a solution of (II_ε):

$$J_\varepsilon = \int_\Omega \varepsilon X_\varepsilon \left[\varphi (\mu u_\varepsilon + \frac{1}{\varepsilon} W_\varepsilon u_\varepsilon - f) + \sum_{i,j} a^*_{ij_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\right] dx .$$

Going to the limit (as $\varepsilon \to 0$)

$$\lim_{\varepsilon \to 0} J_\varepsilon = J = M(WX) \int_\Omega u\varphi dx ,$$

that is to say

(1.10) $$- \sum_i \alpha_{i_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} \xrightarrow[(\varepsilon \to 0)]{w - L^2} M(WX) u ;$$

from (1.8), (1.9), (1.10) we get

$$\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon \xrightarrow[\varepsilon \to 0]{w - H^{-1}} \sum_i M(\lambda_i) \frac{\partial u}{\partial x_i} + M(WX)u \quad (1.11)$$

c) In order to compute the limit equation and to avoid computing twice the same limits, from (1.8), we write the equation (II_ε) in the following form:

$$\mu u_\varepsilon - \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_j}\right) + \sum_i \frac{\partial}{\partial x_i}(\alpha_{i_\varepsilon} u_\varepsilon) = f + \sum_i \alpha_{i_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} .$$

Let us define the family of uniformly elliptic operators $(B^\varepsilon)_{\varepsilon>0}$:

$$B^\varepsilon v = \mu v - \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij_\varepsilon} \frac{\partial v}{\partial x_j}\right) + \sum_i \frac{\partial}{\partial x_i}(\alpha_{i_\varepsilon} v) . \quad (1.12)$$

Our problem reduces computing the limit B of the sequence $(B^\varepsilon)_{\varepsilon>0}$ in variational sense (that is to say the homogenized operator of the $(B^\varepsilon)_{\varepsilon>0}$); since $f + \sum \alpha_{i_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}$ converge in weak-L^2 (hence in strong H^{-1}) to $f - M(WX)u$ the limit equation will be:

$$B(u) + M(WX)u = f . \quad (1.13)$$

d) Let us compute B: given g in $L^2(\Omega)$, let v_ε be the solution of

$$\mu v_\varepsilon - \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij_\varepsilon} \frac{\partial v_\varepsilon}{\partial x_j}\right) + \sum_i \frac{\partial}{\partial x_i}(\alpha_{i_\varepsilon} v_\varepsilon) = g \qquad v_\varepsilon|_{\partial\Omega} = 0 . \quad (1.14)$$

Let us introduce

$$\xi_i^\varepsilon = \sum_j a_{ij_\varepsilon} \frac{\partial v_\varepsilon}{\partial x_j}$$

and let us extract weakly converging subsequences $\xi_i^\varepsilon \xrightarrow{w - L^2} \xi_i$, $v_\varepsilon \xrightarrow{w - H_0^1} v$. The limit equation will be

$$\mu v - \sum_i \frac{\partial \xi_i}{\partial x_i} + \sum_i M(\alpha_i) \frac{\partial v}{\partial x_i} = g . \quad (1.15)$$

The problem is to identify the ξ_i: just like in the construction of the homogenized operator A of the family $(A^\varepsilon)_{\varepsilon>0}$ let us introduce X^i solution of

$$\left|\begin{array}{l} A^*(X^i - x_i) = 0 \ \text{ and } \ w^i = x_i - X^i \\ X^i \ \text{Y-periodic} . \end{array}\right. \quad (1.16)$$

Given $\varphi \in C_0^\infty(\omega)$ let us multiply (1.14) by $\varepsilon w_\varepsilon^i \varphi$ and integrate over Ω: let

$$K_\varepsilon = \int_\Omega \sum_j \xi_j^\varepsilon \frac{\partial \varphi}{\partial x_j} \varepsilon w_\varepsilon^i dx + \int_\Omega \sum_j \frac{\partial v_\varepsilon}{\partial x_j} \left(\sum_k a_{kj_\varepsilon} \frac{\partial}{\partial x_k} (\varepsilon w_\varepsilon^i)\right)\varphi dx - \int_\Omega \sum_j \alpha_{j_\varepsilon} v_\varepsilon \frac{\partial}{\partial x_j} (\varepsilon w_\varepsilon^i \varphi) dx ,$$

$$K_\varepsilon = a_\varepsilon + b_\varepsilon + c_\varepsilon .$$

Then the identity we get can be written

(1.17) $$K_\varepsilon = (g - \mu v_\varepsilon, \varepsilon w_\varepsilon^i \varphi) .$$

Let us compute the limits of a_ε, b_ε, c_ε: since

$$\varepsilon w_\varepsilon^i = x_i - \varepsilon X_\varepsilon^i, \quad \varepsilon w_\varepsilon^i \xrightarrow[(\varepsilon \to 0)]{L^2} x_i$$

which implies

(1.18) $$a_\varepsilon \xrightarrow[(\varepsilon \to 0)]{} \int_\Omega \sum_j \xi_j \frac{\partial \varphi}{\partial x_j} x_i dx .$$

Integrating by parts,

$$b_\varepsilon = - \int_\Omega \varphi v_\varepsilon \sum_{k,j} \frac{\partial}{\partial x_j} \left(a_{kj_\varepsilon} \frac{\partial}{\partial x_k} (\varepsilon w_\varepsilon^i)\right) dx - \int_\Omega v_\varepsilon \sum_j \frac{\partial \varphi}{\partial x_j} \left(\sum_k a_{kj_\varepsilon} \frac{\partial}{\partial x_k} (\varepsilon w_\varepsilon^i)\right) dx .$$

From (1.16) $A^* w^i = 0$, and $A_\varepsilon^*(\varepsilon w_\varepsilon^i) = 0$, therefore

$$b_\varepsilon = - \int_\Omega v_\varepsilon \sum_j \frac{\partial \varphi}{\partial x_j} \left(\sum_k a_{kj_\varepsilon} \frac{\partial}{\partial x_k} (\varepsilon w_\varepsilon^i)\right) dx .$$

Since

$$\frac{\partial}{\partial x_k} (\varepsilon w_\varepsilon^i) = \delta_{ik} - \left(\frac{\partial X^i}{\partial x_k}\right)_\varepsilon ,$$

b_ε may be written

$$b_\varepsilon = - \int_\Omega v_\varepsilon \sum_j \left(a_{ij} - \sum a_{kj} \frac{\partial X^i}{\partial x_k}\right)_\varepsilon \frac{\partial \varphi}{\partial x_j} dx$$

and

(1.19) $$b_\varepsilon \xrightarrow[\varepsilon \to 0]{} - \sum_j M\left(a_{ij} - \sum_k a_{kj} \frac{\partial X^i}{\partial x_k}\right) \int_\Omega v \frac{\partial \varphi}{\partial x_j} dx .$$

Let us look finally to c_ε:

$$c_\varepsilon = - \int_\Omega \sum_j \alpha_{j_\varepsilon} v_\varepsilon \frac{\partial}{\partial x_j} (\varepsilon w_\varepsilon^i \varphi) dx ,$$

$$c_\varepsilon = -\int_\Omega \sum_j \alpha_{j_\varepsilon} v_\varepsilon \varepsilon w_\varepsilon^i \frac{\partial\varphi}{\partial x_j}\,dx - \int_\Omega \alpha_{i_\varepsilon} v_\varepsilon \varphi dx + \int_\Omega \sum_j \alpha_{j_\varepsilon} v_\varepsilon \left(\frac{\partial X^i}{\partial x_j}\right)_\varepsilon \varphi dx ,$$

(1.20) $$c_\varepsilon \xrightarrow[(\varepsilon \to 0)]{} - \sum_j M(\alpha_j) \int_\Omega v.x_i.\frac{\partial\varphi}{\partial x_j}\,dx - M(\alpha_i) \int_\Omega v\varphi dx + \sum_j M\left(\alpha_j \frac{\partial X^i}{\partial x_j}\right) \int_\Omega v\varphi dx .$$

From (1.17), (1.18), (1.19), (1.20) it follows

(1.21) $$\int_\Omega \sum_j \xi_j \frac{\partial\varphi}{\partial x_j} x_i dx - \sum_j M\left(a_{ij} - \sum_k a_{kj} \frac{\partial X^i}{\partial x_k}\right) \int_\Omega v \frac{\partial\varphi}{\partial x_j}\,dx - \sum_j M(\alpha_j) \int_\Omega v \frac{\partial}{\partial x_j}(x_i\varphi)dx$$

$$+ \sum_j M\left(\alpha_j \frac{\partial X^i}{\partial x_j}\right) \int_\Omega v\varphi dx = \int_\Omega (g - \mu v) x_i \varphi dx .$$

On the other hand, multiplying (1.14) by $x_i\varphi$, integrating over Ω, and going to the limit as $\varepsilon \to 0$, we get:

(1.22) $$\int_\Omega \sum_j \xi_j \frac{\partial}{\partial x_j}(x_i\varphi)dx - \sum_j M(\alpha_j) \int_\Omega v \frac{\partial}{\partial x_j}(x_i\varphi)dx = \int_\Omega (g - \mu v)\varphi x_i dx .$$

From (1.21) and (1.22) we get

(1.23) $$\int_\Omega \xi_i \varphi dx = -\sum_j M\left(a_{ij} - \sum a_{kj} \frac{\partial X^i}{\partial x_k}\right) \int_\Omega v \frac{\partial\varphi}{\partial x_j}\,dx + \sum_j M\left(\alpha_j \frac{\partial X^i}{\partial x_j}\right) \int_\Omega v\varphi dx .$$

Since this is true for any $\varphi \in C_0^\infty(\Omega)$ we get the following equality (in distribution sense):

(1.24) $$\xi_i = \sum_j M\left(a_{ij} - \sum_k a_{kj} \frac{\partial X^i}{\partial x_k}\right) \frac{\partial v}{\partial x_j} + M\left(\sum_j \alpha_j \frac{\partial X^i}{\partial x_j}\right) v .$$

From (1.15) the limit equation is

(1.25) $$\mu v - \sum_{i,j} M\left(a_{ij} - \sum a_{kj} \frac{\partial X^i}{\partial x_k}\right) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_i M\left(\alpha_i - \sum_j \alpha_j \frac{\partial X^i}{\partial x_j}\right) \frac{\partial v}{\partial x_i} = f ,$$

(1.26) $$\mathcal{B}v = \mathcal{A}v + \sum_i M\left(\alpha_i - \sum_j \alpha_j \frac{\partial X^i}{\partial x_j}\right) \frac{\partial v}{\partial x_i} .$$

e) From (1.13) the limit equation is

(II) $$\mu u + \mathcal{A}(u) + \sum_i M\left(\alpha_i - \sum_j \alpha_j \frac{\partial X^i}{\partial x_j}\right) \frac{\partial u}{\partial x_i} + M(WX)u = f \text{ on } \Omega; \quad u|_{\partial\Omega} = 0 .$$

Let us look in detail to the coefficients of the first order term: by definition, $A^* X^i = -\sum \frac{\partial}{\partial x_k} (a^*_{ki})$; let us multiply by X and integrate by parts: we get

$$\int_Y \sum a^*_{kj} \frac{\partial X^i}{\partial x_j} \frac{\partial X}{\partial x_k} dx = \int_Y \sum_k a^*_{ki} \frac{\partial X}{\partial x_k} dx ;$$

on the other hand

$$\int_Y \sum_j \alpha_j \frac{\partial X^i}{\partial x_j} dx = \int_Y \sum a^*_{jk} \frac{\partial X^i}{\partial x_j} \frac{\partial X}{\partial x_k} .$$

Since

$$\int_Y \alpha_i dx = \int_Y \sum a^*_{ik} \frac{\partial Y}{\partial x_k} dx$$

we see that all these quantities are equal when the coefficients are symmetric; in that case the limit equation reduces to

$(II)_{bis}$ $$\mu u + A(u) + M(WX)u = f \text{ on } \Omega ; \quad u|_{\partial\Omega} = 0 .$$

This is also the case if the coefficients are constant $(M(\alpha_i) \equiv 0)$.

<u>Remark 4</u>. We may write the equation (II) in the following form

$$A^\varepsilon u_\varepsilon = f - \frac{1}{\varepsilon} W_\varepsilon u_\varepsilon - \mu u_\varepsilon .$$

We know that $A^\varepsilon \to A$ in variational sense (or in G-sense) which clearly implies that:

$$\forall (v_\varepsilon, g_\varepsilon) \in A^\varepsilon \quad v_\varepsilon \xrightarrow{w - H^1_0} v, \quad g_\varepsilon \xrightarrow{s - H^{-1}} g \Longrightarrow (u,g) \in A .$$

But we cannot use this argument in order to go to the limit since the sequence $(f - \frac{1}{\varepsilon} W_\varepsilon u_\varepsilon - \mu u_\varepsilon)_{\varepsilon>0}$ converge only in weak-H^{-1} its limit being equal to $f - \mu u - \sum M(\alpha_i) \frac{\partial u}{\partial x_i} - M(WX)u$.

Actually, the limit equation is not

$$\mu u + A(u) + \sum M(\alpha_i) \frac{\partial u}{\partial x_i} + M(WX)u = f !$$

So, the arguments developed in the proof of Theorem 1, justify and extend to the non-symmetric case the result of B.L.P. [2] (Theorem 12.6).

Another way of looking at and extending the previous results is to consider them as homogenization problems with lower order terms. That's what we are going to look at in the next paragraph.

1.4. Homogenization with lower order terms.

We are going to prove the following statement:

Theorem 2. Let u be the solution (μ is taken large enough, all the coefficients are Y-periodic, the $(a_{ij})_{ij}$ are uniformly elliptic) of the following problem:

$$(\mathrm{III})_\varepsilon \quad \mu u_\varepsilon - \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_j}\right) + \sum_i \frac{\partial}{\partial x_i}(\gamma_{i_\varepsilon} u_\varepsilon) + \sum_i \beta_{i_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i} = f \ \text{ on } \Omega, \quad u_\varepsilon|_{\partial\Omega} = 0 .$$

When ε goes to zero, u_ε converge weakly to u in H_0^1, where u is the solution of:

$$(\mathrm{III}) \quad \left|\begin{array}{l} \mu u + \mathcal{A}(u) + \sum_i M\left(\gamma_i - \sum_j \gamma_j \frac{\partial X^i}{\partial x_j}\right)\frac{\partial u}{\partial x_i} + \sum_i M\left(\beta_i + \sum_j a^*_{ij}\frac{\partial \beta}{\partial x_j}\right)\frac{\partial u}{\partial x_i} - M\left(\sum_i \gamma_i \frac{\partial \beta}{\partial x_i}\right) u = f \\ u|_{\partial\Omega} = 0 \end{array}\right.$$

where we denote:

$\mathcal{A}$ is the classical homogenized operator of the $(A^\varepsilon)_{\varepsilon>0}$ (cf. 1.5)

the X^i are defined in (1.5)

β is defined by:

$$(1.27) \qquad \left|\begin{array}{l} A^*\beta = \sum_j \frac{\partial \beta_j}{\partial x_j} \\ \beta \ Y\text{-periodic} . \end{array}\right.$$

Proof of Theorem 2. From (1.12) (III_ε) can be written:

$$(1.28) \qquad \mathcal{B}^\varepsilon u_\varepsilon + \sum_i \beta_{i_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i} = f .$$

The only problem is to compute the weak limit in $L^2(\Omega)$ (let us call it η) of the sequence $\left(\sum_i \beta_{i_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i}\right)_{\varepsilon>0}$; the limit equation will be

$$(1.29) \qquad \mathcal{B}u + \eta = f$$

with $\mathcal{B}$ given by (1.26):

$$(1.26) \qquad \mathcal{B}u = \mu u + \mathcal{A}(u) + \sum_i M\left(\gamma_i - \sum_j \gamma_j \frac{\partial X^i}{\partial x_j}\right)\frac{\partial u}{\partial x_i} .$$

From (1.27) $A^*_\varepsilon(\varepsilon\beta_\varepsilon) = \sum_j \frac{\partial}{\partial x_j}(\varepsilon\beta_{j_\varepsilon})$; let us multiply (1.27) by φu_ε and integrate by parts:

$$\int_\Omega \sum_{i,j} a^*_{ij} \frac{\partial}{\partial x_j}(\varepsilon\beta_\varepsilon) \frac{\partial}{\partial x_i}(\varphi u_\varepsilon)dx = -\sum_i \int_\Omega \beta_{i_\varepsilon} \frac{\partial}{\partial x_i}(\varphi u_\varepsilon)dx \;;$$

this implies

$$-\int_\Omega \left(\sum_i \beta_{i_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\right)\varphi dx = \int_\Omega \sum_i \beta_{i_\varepsilon} u_\varepsilon \frac{\partial\varphi}{\partial x_i} dx + \int_\Omega \varphi \sum_j \frac{\partial}{\partial x_j}(\varepsilon\beta_\varepsilon)\left(\sum_i a^*_{ij_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\right)dx$$

$$+ \int_\Omega \sum \left(a^*_{ij} \frac{\partial\beta}{\partial x_j}\right)_\varepsilon u_\varepsilon \frac{\partial\varphi}{\partial x_i} dx \;,$$

(1.30)
$$-\int_\Omega \left(\sum_i \beta_{i_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\right)\varphi dx = a_\varepsilon + b_\varepsilon + c_\varepsilon \;.$$

Clearly

(1.31)
$$a_\varepsilon = \int_\Omega \sum_i \beta_{i_\varepsilon} u_\varepsilon \frac{\partial\varphi}{\partial x_i} dx \xrightarrow[(\varepsilon\to 0)]{} \sum_i M(\beta_i) \int_\Omega u \frac{\partial\varphi}{\partial x_i} dx \;,$$

(1.32)
$$c_\varepsilon = \int_\Omega \left(\sum a^*_{ij} \frac{\partial\beta}{\partial x_j}\right)_\varepsilon u_\varepsilon \frac{\partial\varphi}{\partial x_i} dx \xrightarrow[(\varepsilon\to 0)]{} \sum_i M\left(\sum_j a^*_{ij} \frac{\partial\beta}{\partial x_j}\right) \int_\Omega u \frac{\partial\varphi}{\partial x_i} dx \;.$$

Let us consider

$$b_\varepsilon = \int_\Omega \sum_j \frac{\partial}{\partial x_j}(\varepsilon\beta_\varepsilon)\left(\sum_i a^*_{ij_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\right)\varphi dx$$

and integrate by parts

$$b_\varepsilon = -\int_\Omega \varepsilon\beta_\varepsilon\left[\sum_{i,j} \frac{\partial}{\partial x_j}\left(a^*_{ij_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\right)\varphi + \sum_{i,j} a^*_{ij_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial\varphi}{\partial x_j}\right]dx \;.$$

Using that u_ε satisfies (III)

$$b_\varepsilon = -\int_\Omega \varepsilon\beta_\varepsilon\left[\varphi\left(\mu u_\varepsilon + \sum \frac{\partial}{\partial x_i}(\gamma_{i_\varepsilon} u_\varepsilon) + \sum \beta_{i_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} - f\right) + \sum_{i,j} a^*_{ij_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial\varphi}{\partial x_j}\right]dx \;.$$

Clearly,

$$\lim_{\varepsilon\to 0} b_\varepsilon = \lim_{\varepsilon\to 0}\left\{-\int_\Omega \varepsilon\beta_\varepsilon \varphi \sum \frac{\partial}{\partial x_i}(\gamma_{i_\varepsilon} u_\varepsilon)dx\right\}$$

$$= \lim_{\varepsilon\to 0} \int_\Omega \sum_i \gamma_{i_\varepsilon} u_\varepsilon\left(\varepsilon\beta_\varepsilon \frac{\partial\varphi}{\partial x_i} + \left(\frac{\partial\beta}{\partial x_i}\right)_\varepsilon \varphi\right)dx$$

and

$$(1.33) \qquad b_\varepsilon \xrightarrow[(\varepsilon \to 0)]{} M\left(\sum_i \gamma_i \frac{\partial\beta}{\partial x_i}\right) \int_\Omega u\varphi dx \ .$$

From (1.30), (1.31), (1.32), (1.33) we get:

$$(1.34) \qquad -\int_\Omega \eta\varphi dx = \sum_i M(\beta_i) \int_\Omega u \frac{\partial\varphi}{\partial x_i} dx + M\left(\sum \gamma_i \frac{\partial\beta}{\partial x_i}\right) \int u\varphi dx + \sum M\left(\sum a^*_{ij} \frac{\partial\beta}{\partial x_j}\right) \int_\Omega u \frac{\partial\varphi}{\partial x_i} dx \ ,$$

$$(1.35) \qquad \eta = \sum_i M\left(\beta_i + \sum_j a^*_{ij} \frac{\partial\beta}{\partial x_j}\right) \frac{\partial u}{\partial x_i} - M\left(\sum \gamma_i \frac{\partial\beta}{\partial x_i}\right) u \ .$$

From (1.26), (1.29) and (1.35) the limit equation is:

$$(\mathrm{III}) \qquad \left| \begin{array}{l} \mu u + A(u) + \sum_i \left\{ M\left(\gamma_i - \sum_j \gamma_j \frac{\partial\chi^i}{\partial x_j}\right) + M\left(\beta_i + \sum a^*_{ij} \frac{\partial\beta}{\partial x_j}\right) \right\} \frac{\partial u}{\partial x_i} - M\left(\sum_i \gamma_i \frac{\partial\beta}{\partial x_i}\right) u = f \\ u|_{\partial\Omega} = 0 \ . \end{array} \right.$$

Remark 5.

a) When $\beta_i = -\gamma_i$ with $\gamma_i = \sum_k a^*_{ik} \frac{\partial\chi}{\partial x_k}$ (which was denoted α_i, cf. (1.7), Thm.1) we get

$$\sum \frac{\partial\beta_i}{\partial x_i} = -\sum \frac{\partial}{\partial x_i} \left(a^*_{ik} \frac{\partial\chi}{\partial x_k}\right) = A^*\chi$$

and, in (1.27), we can take $\beta = \chi$. It follows that

$$\beta_i + \sum_j a^*_{ij} \frac{\partial\beta}{\partial x_j} = -\sum a^*_{ik} \frac{\partial\chi}{\partial x_k} + \sum a^*_{ik} \frac{\partial\beta}{\partial x_k} = 0 \ ,$$

and

$$-M\left(\sum_i \gamma_i \frac{\partial\beta}{\partial x_i}\right) = -\frac{1}{|Y|} \int_Y \sum_{i,k} a^*_{ik} \frac{\partial\chi}{\partial x_k} \frac{\partial\chi}{\partial x_i} dx$$

is equal to $M(W\chi)$ since

$$M(W\chi) = -\frac{1}{|Y|} \int_\Omega A^*\chi\, \chi dy = -\frac{1}{|Y|} \int_\Omega \sum_{i,k} a^*_{ik} \frac{\partial\chi}{\partial x_k} \frac{\partial\chi}{\partial x_i} dx \ .$$

So, formula (III) reduces to (II) and we refind Theorem 1.

b) When $\gamma_i = 0$, we obtain Theorem 13.1 of B.L.P. [2].

1.5. Study of the general problem; conjecture.

Let us consider the problem (P); how to compute the limit in variational sense of the sequence F_ε where

$$F_\varepsilon(u) = \int_\Omega f(\tfrac{x}{\varepsilon}, u, Du)dx$$

with f Y-periodic in x, convex in (u,Du) and $\lambda_0|z|^p \le f(x,\xi,z) \le \Lambda_0(1 + |\xi|^p + |z|^p)$, $p > 1$. From the general compactness Theorem A, one can extract a subsequence F_{ε_k} and find a normal converse integrand g such that

$$\forall\Omega \in A_{p_n} \quad \forall u \in W_0^{1,p} \quad F_0(u,\Omega) = \Gamma^-(w - W_0^{1,p})\lim F_{\varepsilon_k}(u,\Omega)$$

where $F_0(u,\Omega) = \int_\Omega g(x,u,Du)dx$ (A_{p_n} is the family of bounded open sets in $\mathbb{R}^n$).

The problem is to identify g: in the three cases we already saw (f independent of ξ, f independent of z, f quadratic in (ξ,z)), the integrand g does not depend on x. Actually, this is also true in the general case:

<u>Proposition 1</u>. The integrand g is independent of x.

<u>Proof of Proposition 1</u>. Following the proof of [1] Theorem 2.1, there exist a convex integrand f_0 such that:

$$\forall(\xi z) \in \mathbb{R}\times\mathbb{R}^n \quad \forall\Omega \in A_{p_n} \quad F_0(\xi + \langle z,\cdot\rangle,\Omega) = \int_\Omega f_0(x,\xi,z)dx$$

and

$$\forall u \in W_0^{1,p}(\Omega) \quad F_0(u,\Omega) = \int_\Omega f_0(x,u(x) - \langle Du(x),x\rangle, Du(x))dx .$$

Let

$$u_{\varepsilon_k} \xrightarrow[k\to+\infty]{L^1(\Omega)} \xi + \langle z,\cdot\rangle$$

such that

$$F_0(\xi + \langle z,\cdot\rangle,\Omega) = \lim_{k\to+\infty} F_{\varepsilon_k}(u_{\varepsilon_k},\Omega) ,$$

$$F_{\varepsilon_k}(u_{\varepsilon_k},\Omega) = \int_\Omega f(\tfrac{x}{\varepsilon_k}, u_{\varepsilon_k}, Du_{\varepsilon_k})dx ;$$

since $f(\cdot,\xi,z)$ is Y-periodic

$$= \int_{\Omega+n_k^i\varepsilon_k} f(\tfrac{x}{\varepsilon_k}, u_{\varepsilon_k}(x - n_k^i\varepsilon_k), Du_{\varepsilon_k}(x - n_k^i\varepsilon_k)dx .$$

Let $x_0 \in \mathbb{R}^n$ be fixed and take n_k^i such that $n_k^i\varepsilon_k \le x_0^i \le (n_k^i + 1)\varepsilon_k$. Therefore, $|x_0^i - n_k^i\varepsilon_k| \le \varepsilon_k$; take $\Omega = Y_\varepsilon + x_1$ where $\varepsilon > 0$ and $x_1 \in \mathbb{R}^n$ are fixed. From continuity assumptions on the functionals F_ε (which follows from the convexity and uniform boundedness assumptions)

$$\left|F_{\varepsilon_k}(u_{\varepsilon_k}, Y_\varepsilon + x_1) - F_{\varepsilon_k}(v_k, Y_\varepsilon + x_1 + x_0)\right| \to 0$$

where v_k is equal to $u_{\varepsilon_k}(\cdot - n_k^i \varepsilon_k)$ on $\Omega + n_k^i \varepsilon_k$ and is extended with uniformly bounded derivatives to $\Omega + x_0$.

Since v_k converge to $(\xi - (z, x_0)) + (z, \cdot)$ in $L^1(Y_\varepsilon + x_0 + x_1)$

$$F_0(\xi - (z, x_0) + (z, \cdot), Y_\varepsilon + x_0 + x_1) \leq \liminf F_{\varepsilon_k}(v_k, Y_\varepsilon + x_1 + x_0)$$

$$\leq \liminf F_{\varepsilon_k}(u_{\varepsilon_k}, Y_\varepsilon + x_1) .$$

Therefore,

$$\forall \varepsilon > 0 \quad \int_{Y_\varepsilon + x_0 + x_1} f_0(x, \xi - (z, x_0), z)dx \leq \int_{Y_\varepsilon + x_1} f_0(x, \xi, z)dx .$$

Making ε go to zero, this implies

$$f_0(x_0 + x_1, \xi - (z, x_0), z) \leq f_0(x_1, \xi, z)$$

that is to say

(1.36) $$\forall (x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n \quad f_0(x_0 + x_1, \xi, z) \leq f_0(x_1, \xi + (z, x_0), z) .$$

Writing $x_1 = (x_1 + x_0) - x_0$ and applying once more (1.36), we get

(1.37) $$\forall (x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n \quad f_0(x_1, \xi + (z, x_0), z) \leq f_0(x_1 + x_0, \xi, z) .$$

From (1.36) and (1.37) it follows

(1.38) $$\forall (x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n \quad \forall \xi \in \mathbb{R} \quad \forall z \in \mathbb{R}^n \quad f_0(x_0 + x_1, \xi, z) = f_0(x_1, \xi + (z, x_0), z)$$

and taking $x_1 = 0$

(1.39) $$f_0(x, \xi, z) = f_0(0, \xi + (z, x), z) ;$$

this implies

$$F_0(u, \Omega) = \int_\Omega f_0(x, u(x) - (Du(x), x), Du(x))dx = \int_\Omega f_0(0, u(x), Du(x))dx \quad \text{i.e.} \quad g(x, \xi, z) = f_0(0, \xi, z) .$$

<u>Remark 6</u>. It should be interesting to get a general answer to the problem: to know if, under periodicity conditions on x and convexity on (ξ, z) of f, the sequence

$F_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, u, Du)dx$ converge and what is its limit equal to. It seems reasonable to conjecture such a result. Let us consider now the problem of the identification of f:

If the integrand f is independent of z:

$$F_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, u)dx$$

the Γ^- limit of F_ε is equal to (cf. [5])

$$F_0(u) = \int_\Omega f_0(u)dx \quad \text{with} \quad f_0(\xi) = \frac{1}{|Y|}\int_Y f(y,\xi)dy .$$

If the integrand f is independent of ξ

$$F_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, Du(x))dx$$

the Γ^- limit of F_ε is equal to (cf. [3])

$$F_0(u) = \int_\Omega f_0(Du)dx \quad \text{with} \quad f_0(z) = \min_{u \in W_Y} \frac{1}{|Y|}\int_Y f(y, Du + z)dy$$

where we denote W_Y the space of the Y-periodic functions in $W^{1,p}$.

A natural conjecture concerning the case where f depends on ξ and z would be that:

$$F_0(u) = \int_\Omega f_0(u, Du)dx \quad \text{with} \quad f_0(\xi, z) = \min_{u \in W_Y} \frac{1}{|Y|}\int_Y f(y, \xi, Du + z)dy .$$

This formula is correct in the two preceding cases and in fact that is the one conjectured by Bensoussan, Lions and Papanicolau [2], Remark 17.7.

We are going to prove that unfortunately it is not correct in the general case; in order to give a <u>counter example</u> we shall use the explicit computation we made of the limit functional in the quadratic case: Let us consider

$$F_\varepsilon(u) = \int_\Omega \left\{\frac{1}{2}\sum_{i,j} a_{ij_\varepsilon} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_i b_{i_\varepsilon} u \frac{\partial u}{\partial x_i} + \frac{1}{2} c_\varepsilon u^2\right\}dx .$$

That is to say

$$F_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, u, Du)dx$$

with

(1.37) $$f(y,\xi,z) = \frac{1}{2}\sum a_{ij}(y) z_i z_j + \sum b_i(y)\xi z_i + \frac{1}{2} c(y)\xi^2 .$$

From Corollary 1, we know that

$$F_0 = \Gamma^-(w - H_0^1) \lim_{\varepsilon\to 0} F_\varepsilon$$

is equal to:

$$F_0(u) = \int_\Omega \left\{\frac{1}{2} \sum_{i,j} q_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{1}{2} M(c + WX)u^2\right\} dx .$$

That is to say

$$F_0(u) = \int_\Omega f_0(u,Du)\,dx$$

with

(1.38) $$f_0(\xi,z) = \frac{1}{2} \sum_{i,j} q_{ij} z_i z_j + \frac{1}{2} M(c + WX)\xi^2$$

where

(1.39) $$q_{ij} = M\left(a_{ij} - \sum_k a_{kj} \frac{\partial X^i}{\partial x_k}\right) \qquad A(X^i) = A(x_i) = -\sum_k \frac{\partial}{\partial x_k} (a_{ki})$$

and

(1.40) $$W = -\sum_{i=1}^n \frac{\partial b_i}{\partial x_i}, \qquad \begin{vmatrix} A(X) + W = 0 \\ X \text{ Y-periodic} . \end{vmatrix}$$

So, let us compute

$$I = \min_{w \in W_Y} \frac{1}{|Y|} \int_Y f(y,\xi,Du + z)\,dy$$

and compare to f_0. Let $u_{\xi,z}$ be a minimizing point for I: it satisfies

(1.41) $$-\sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u_{\xi,z}}{\partial x_j}\right) - \sum_{i,j} \frac{\partial}{\partial x_j} (a_{ij} z_j) - \xi \sum_i \frac{\partial}{\partial x_i} b_i = 0$$

equivalently

$(1.41)_{bis}$ $$A(u_{\xi,z} + \langle z,\cdot\rangle) = -\xi W .$$

From (1.40)

(1.42) $$A(u_{\xi,z} + \langle z,\cdot\rangle) = A(\xi X) .$$

From (1.42), we cannot conclude directly, since $u_{\xi,z} + \langle z,\cdot\rangle$ is not Y-periodic; so we remark that

(1.39) $$AX^i = Ax_i \Longrightarrow A(\langle z,\cdot\rangle) = A\left(\sum_i z_i X^i\right) .$$

Therefore

$$A(u_{\xi,z} + \sum_i z_i X^i) = A(\xi X)$$

and now, we can conclude since ξX and $u_{\xi,z} + \sum z_i X^i$ are Y-periodic:

$$u_{\xi,z} + \sum z_i X^i = \xi X \quad \text{(up to a constant)} ,$$

which implies

$$Du_{\xi,z} + \sum z_i DX^i = \xi DX \tag{1.43}$$

and

$$Du_{\xi,z} + z = \xi DX - \sum_k z_k DX^k + z . \tag{1.43)$_{bis}$ (}$$

It follows that:

$$I = \frac{1}{|Y|} \int_Y \left[\frac{1}{2} \sum_{i,j} a_{ij}\left(\xi \frac{\partial X}{\partial x_i} - \sum_k z_k \frac{\partial X^k}{\partial x_i} + z_i\right)\left(\xi \frac{\partial X}{\partial x_j} - \sum_\ell z_\ell \frac{\partial X^\ell}{\partial x_j} + z_j\right)\right.$$

$$\left. + \sum_i b_i \xi\left(\xi \frac{\partial X}{\partial x_i} - \sum z_k \frac{\partial X^k}{\partial x_i} + z_i\right) + \frac{1}{2} c\xi^2\right]dx .$$

Let us order the terms with respect to ξ^p, $p = 0,1,2$:

$$I = (\alpha + \beta\xi + \gamma\xi^2)$$

with

$$\alpha = \frac{1}{|Y|} \int_Y \frac{1}{2}\left(\sum_{i,j} a_{ij}z_i z_j + \sum_{i,j,k,\ell} a_{ij}z_k z_\ell \frac{\partial X^k}{\partial x_i} \frac{\partial X^k}{\partial x_\ell} - \sum_{i,j,k} a_{ij} \frac{\partial X^k}{\partial x_i} z_k z_j - \sum_{i,j,\ell} a_{ij} \frac{\partial X^\ell}{\partial x_j} z_i z_\ell\right)dx,$$

$$\beta = \frac{1}{|Y|} \int_Y \frac{1}{2} \left(- \sum_{i,j,k} a_{ij} \frac{\partial X}{\partial x_i} z_k \frac{\partial X^k}{\partial x_j} - \sum_{i,j,k} a_{ij} \frac{\partial X}{\partial x_j} z_k \frac{\partial X^k}{\partial x_i} + \sum a_{ij} \frac{\partial X}{\partial x_i} z_j + \sum a_{ij} \frac{\partial X}{\partial x_j} z_i\right)dx$$

$$+ \frac{1}{|Y|} \int_Y \sum_i b_i\left(z_i - \sum_k z_k \frac{\partial X^k}{\partial x_i}\right)dx ,$$

$$\gamma = \frac{1}{|Y|} \int_Y \left(\frac{1}{2} \sum_{i,j} a_{ij} \frac{\partial X}{\partial x_i} \frac{\partial X}{\partial x_j} + \sum_i b_i \frac{\partial Y}{\partial x_i} + \frac{1}{2} c\right)dx .$$

<u>Computation of α:</u>

$$\alpha = \frac{1}{2} M\left(\sum_{i,j} a_{ij}z_i z_j + \sum_{k,\ell,i,j} a_{ij}z_k z \frac{\partial X^k}{\partial x_i} \frac{\partial X^k}{\partial x} - 2 \sum_{i,j,k} a_{ij} \frac{\partial X^k}{\partial x_i} z_k z_j\right) ,$$

$$\alpha = \frac{1}{2} M\left(\sum_{i,j}\left[a_{ij} + \sum_{k,\ell} a_{k\ell} \frac{\partial X^i}{\partial x_k} \frac{\partial X^j}{\partial x_\ell} - 2 \sum_k a_{kj} \frac{\partial X^i}{\partial x_k}\right] z_i z_j\right) .$$

From (1.39)

$$AX^i = -\sum \frac{\partial}{\partial x_k} (a_{ki})$$

which implies

$$(AX^i, X^j) = \int_Y \sum_k a_{ki} \frac{\partial X^j}{\partial x_k} dx = \int_Y \sum a_{kj} \frac{\partial X^i}{\partial x_k} dx \ ;$$

on the other hand

$$(AX^i, X^j) = \int_Y \sum a_{k\ell} \frac{\partial X^i}{\partial x_k} \frac{\partial X^j}{\partial x_\ell} dx \ .$$

It follows that

$$\alpha = \frac{1}{2} M\left(\sum_{i,j} \left(a_{ij} - \sum_k a_{kj} \frac{\partial X^i}{\partial x_k}\right)\right) z_i z_j = \frac{1}{2} \sum_{i,j} q_{ij} z_i z_j \ .$$

<u>Computation of γ</u>: From (1.40)

$$AX = \sum \frac{\partial b_i}{\partial x_i}$$

and

$$(AX, X) = -\sum \int_Y b_i \frac{\partial X}{\partial x_i} dx \ ;$$

on the other hand

$$(AX, X) = \sum \int_Y a_{ij} \frac{\partial X}{\partial x_i} \frac{\partial X}{\partial x_j} dx \ .$$

Therefore

$$\gamma = -\frac{1}{2|Y|} \int_Y \sum a_{ij} \frac{\partial X}{\partial x_i} \frac{\partial X}{\partial x_j} dx + \frac{1}{2|Y|} \int_Y c \ dx = \frac{1}{2} M(c + WX) \ .$$

So, up to now the formula is correct.

<u>Computation of β</u>:

$$\beta = M\left(-\sum_{i,j,k} a_{ij} \frac{\partial X}{\partial x_i} \frac{\partial X^k}{\partial x_j} z_k + \sum_{i,j} a_{ij} \frac{\partial X}{\partial x_i} z_j + \sum b_i \left(z_i - \sum_k z_k \frac{\partial X^k}{\partial x_i}\right)\right)$$

$$= M\left(\sum_k \left[-\sum_{i,j} a_{ij} \frac{\partial X}{\partial x_i} \frac{\partial X^k}{\partial x_j} + \sum_i a_{ik} \frac{\partial X}{\partial x_i}\right] z_k + \sum b_i \left(z_i - \sum z_k \frac{\partial X^k}{\partial x_i}\right)\right) .$$

Since

$$AX^k = -\sum_i \frac{\partial}{\partial x_i}(a_{ik}) ,$$

$$\langle AX^k, X\rangle = \int_Y \sum_i a_{ik} \frac{\partial X}{\partial x_i} dx ;$$

on the other hand

$$\langle AX^k, X\rangle = \int_Y \sum a_{ij} \frac{\partial X^k}{\partial x_j} \frac{\partial X}{\partial x_i} dx$$

and β reduces to

$$\beta = M\left(\sum_i b_i\left(z_i - \sum_k z_k \frac{\partial X^k}{\partial x_i}\right)\right) = \sum_i M\left(b_i - \sum_k b_k \frac{\partial X^i}{\partial x_k}\right) z_i$$

which is not in general equal to zero! (In the model example $X^i = 0$ and $b_i = \frac{\partial X}{\partial x_i}$, so $\beta = 0$ and the formula works!)

II. Study of higher order problems

We are going to see in this paragraph that many of the preceding results extend to higher order problems.

2.1. Study of the model problem.

Let us give an energetic proof to the highly oscillating potential problem with the biharmonic operator (cf. B.L.P. [2] for this study through multiscale method).

Proposition 2. Let u_ε (for μ large enough) the solution of

$$\text{(IV)} \qquad \mu u_\varepsilon + \Delta^2 u_\varepsilon + \frac{1}{\varepsilon^2} W_\varepsilon u_\varepsilon = f \quad \text{on } \Omega, \quad u_\varepsilon \in H_0^2(\Omega) .$$

(W is a Y-periodic function with zero mean value.)

When ε goes to zero, u_ε converge weakly in $H_0^2(\Omega)$ to the solution u of

$$\text{(IV)} \qquad \mu u + \Delta^2 u + M(WX)u = f \quad \text{on } \Omega, \quad u \in H_0^2(\Omega)$$

where X is defined by

$$\text{(2.1)} \qquad \left|\begin{array}{l} \Delta^2 X + W = 0 \\ X \text{ is Y-periodic .} \end{array}\right.$$

Proof of Proposition 2. As in the second order case, we remark that u_ε satisfies the Euler equation associated with the functional $F_\varepsilon - (f,\cdot)$ where

$$F_\varepsilon(u) = \int_\Omega \left(\frac{1}{2}\mu u^2 + \frac{1}{2}(\Delta u)^2 + \frac{1}{2\varepsilon^2} W_\varepsilon u\right)dx, \quad u \in H_0^2(\Omega) .$$

Noticing that $\Delta^2(\varepsilon^2 X_\varepsilon) + \frac{1}{\varepsilon^2} W_\varepsilon = 0$, (from 2.1), we can rewrite F_ε:

$$F_\varepsilon(u) = \int_\Omega \left(\frac{1}{2}\mu u^2 + \frac{1}{2}(\Delta u)^2 - 2(\Delta X)_\varepsilon \cdot (u\Delta u + |Du|^2)\right)dx .$$

It follows that for μ large enough F_ε is a convex coercive functional on $H_0^2(\Omega)$ and that u_ε minimizes $F_\varepsilon - (f,\cdot)$ over $H_0^2(\Omega)$.

Moreover, the F_ε being uniformly coercive on $H_0^2(\Omega)$, the $(u_\varepsilon)_{\varepsilon>0}$ remain bounded in $H_0^2(\Omega)$; let $u_\varepsilon \xrightarrow{w - H_0^2} u$.

In order to go to the limit on (IV_ε) we have just to compute the weak-limit in $H^{-2}(\Omega)$ of the sequence $\frac{1}{\varepsilon^2} W_\varepsilon u_\varepsilon$; let us introduce $\varphi \in C_0^\infty(\Omega)$ and look to

$$I_\varepsilon = \int_\Omega \frac{1}{\varepsilon} W_\varepsilon u_\varepsilon \varphi dx = - \int_\Omega \Delta^2(\varepsilon^2 X_\varepsilon) u_\varepsilon \varphi dx ;$$

integrating by parts

$$I_\varepsilon = -\int_\Omega (\Delta X)_\varepsilon [\Delta u_\varepsilon \varphi + 2Du_\varepsilon D\varphi + u_\varepsilon \Delta\varphi] dx \ ;$$

since u_ε converge weakly to u in H_0^2, (and strongly in H_0^1), and $(\Delta X)_\varepsilon \xrightarrow{\sigma(L^\infty, L^1)} 0$

(2.2) $$I_\varepsilon + \int_\Omega (\Delta X)_\varepsilon \Delta u_\varepsilon \varphi dx \xrightarrow[(\varepsilon \to 0)]{} 0 \ ;$$

making another integration by parts and using that u_ε satisfies (IV_ε), we get

$$-\int_\Omega (\Delta X)_\varepsilon \Delta u_\varepsilon \varphi dx = -\int_\Omega \varepsilon^2 X_\varepsilon [\Delta^2 u_\varepsilon \varphi + \Delta u_\varepsilon \Delta\varphi + 2D\varphi D(\Delta u_\varepsilon)] dx \ ,$$

(2.3) $$-\int_\Omega (\Delta X)_\varepsilon \Delta u_\varepsilon \varphi dx = -\int_\Omega \varepsilon^2 X_\varepsilon [(f - \mu u_\varepsilon - \frac{1}{\varepsilon^2} W_\varepsilon u_\varepsilon)\varphi + \Delta u_\varepsilon \Delta\varphi + 2D\varphi D(\Delta u_\varepsilon)] dx \ .$$

Let us consider the last term:

$$J_\varepsilon = \int_\Omega \varepsilon^2 X_\varepsilon D\varphi D(\Delta u_\varepsilon) dx = \sum_i \int_\Omega \varepsilon^2 X_\varepsilon \frac{\partial\varphi}{\partial x_i} \frac{\partial}{\partial x_i} (\Delta u_\varepsilon) dx$$

integrating by parts

(2.4) $$J_\varepsilon = -\sum_i \int \Delta u_\varepsilon \left(\varepsilon^2 X_\varepsilon \frac{\partial\varphi}{\partial x_i} + \varepsilon \left(\frac{\partial X}{\partial x_i}\right)_\varepsilon\right) dx$$ goes to zero as $\varepsilon \to 0$.

From (2.2), (2.3), (2.4) it follows that

$$I_\varepsilon \xrightarrow[\varepsilon \to 0]{} M(WX) \int_\Omega u\varphi dx$$ which means that $\frac{1}{\varepsilon} W_\varepsilon u_\varepsilon \xrightarrow{w - H^{-2}} M(WX)u$

and we finally get the limit equation (IV).

2.2. Homogenization of variational problems for integral functionals, quadratic in (u, Du, D^2u).

The general form of the functionals we shall study in this paragraph is:

(2.5) $$F_\varepsilon(u) = \int_\Omega \left\{\frac{1}{2} \sum a_{ij_\varepsilon} \frac{\partial^2 u}{\partial x_i^2} \frac{\partial^2 u}{\partial x_j^2} + \sum b_{ij_\varepsilon} \frac{\partial^2 u}{\partial x_i^2} \frac{\partial u}{\partial x_j} + \sum c_{i_\varepsilon} \frac{\partial^2 u}{\partial x_i^2} u \right.$$

$$\left. + \frac{1}{2} \sum d_{ij_\varepsilon} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum e_{i_\varepsilon} \frac{\partial u}{\partial x_i} u + \frac{1}{2} f_\varepsilon u^2 \right\} dx$$

where all coefficients are Y-periodic, the (a_{ij}) and (d_{ij}) uniformly coercive:

$$\begin{cases} \sum a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 & (a_{ij} = a_{ji}) \\ \sum d_{ij}\xi_i\xi_j > \mu|\xi|^2 & (d_{ij} = d_{ji}) \\ f \geq \nu > 0 \end{cases}$$

with μ and ν large enough in order the F_ε to be uniformly coercive over $H_0^2(\ )$.

We shall not give a complete answer to the problem which consists knowing if the sequence $(F_\varepsilon)_{\varepsilon>0}$ converge in variational sense (in $\Gamma^-(w - H_0^2)$ sense) and what is its limit equal to. We shall only explain on simpler situations (Theorem 3, Theorem 4, Theorem 5) how to deal with the higher order terms.

The Euler equation corresponding to the critical points of the functional F_ε is:

$$(2.6)\quad \sum \frac{\partial^2}{\partial x_i^2}\left(a_{ij_\varepsilon}\frac{\partial^2 u_\varepsilon}{\partial x_i^2}\right) + \sum_{i,j}\frac{\partial^2}{\partial x_i^2}\left(b_{ij_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_j}\right) - \sum_{i,j}\frac{\partial}{\partial x_j}\left(b_{ij_\varepsilon}\frac{\partial^2 u_\varepsilon}{\partial x_i^2}\right) + \sum \frac{\partial^2}{\partial x_i^2}(c_{i_\varepsilon} u_\varepsilon)$$

$$+ \sum c_{i_\varepsilon}\frac{\partial^2 u_\varepsilon}{\partial x_i^2} - \sum_{i,j}\frac{\partial}{\partial x_i}\left(d_{ij_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_j}\right) - \sum \frac{\partial}{\partial x_i}(e_{i_\varepsilon}u_\varepsilon) + \sum e_{i_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i} + f_\varepsilon u_\varepsilon = f .$$

<u>Theorem 3</u>. Let u_ε be the solution of

$$(V)_\varepsilon \qquad \mu u_\varepsilon + \sum_i \frac{\partial^2}{\partial x_i^2}\left(a_{ij}\frac{\partial^2 u_\varepsilon}{\partial x_j^2}\right) = f \text{ on } \Omega .$$

When $\varepsilon \to 0$, u_ε converge weakly in $H_0^2(\Omega)$ to u solution of

$$(V)\qquad \mu u + \sum q_{ij}\frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} \quad \text{where} \quad q_{ij} = M\left(a_{ij} - \sum_k a_{kj}\frac{\partial^2 \chi^i}{\partial x_k^2}\right)$$

and χ^i satisfies $\left|\begin{array}{l} A\chi^i = \frac{1}{2}A(x_i^2) \quad \text{with} \quad A = \sum \frac{\partial^2}{\partial x_i^2}\left(a_{ij}\frac{\partial^2}{\partial x_j^2}\right) . \\ \chi^i \text{ is } Y\text{-periodic} \end{array}\right.$

<u>Proof of Theorem 3</u>. The $(u_\varepsilon)_{\varepsilon\to 0}$ are bounded in $H_0^2(\Omega)$; let $u_\varepsilon \xrightarrow[\varepsilon \to 0]{w - H_0^2} u$. As in the second order problem, let us introduce

$$\xi_i^\varepsilon = \sum_j a_{ij_\varepsilon}\frac{\partial^2 u_\varepsilon}{\partial x_j^2} .$$

The $(\xi_i^\varepsilon)_{\varepsilon>0}$ are bounded in $L^2(\Omega)$; we can extract subsequences such that $\xi_i^\varepsilon \xrightarrow{w - L^2} \xi_i$. The problem is to identify the ξ_i. Given P an homogenous polynomial of degree two let us introduce χ the solution of

$$(2.7)\qquad \left|\begin{array}{l} A\chi = AP \\ \chi \text{ is } Y\text{-periodic} \end{array}\right.$$

and

(2.8) $$w = p - X ;$$

(2.7), (2.8) imply that $A(\varepsilon^2 w_\varepsilon) = 0$ and $\varepsilon^2 w_\varepsilon = p - \varepsilon^2 X_\varepsilon \xrightarrow{W - H^2} p$.

Let us multiply $(V)_\varepsilon$ by $\varepsilon^2 w_\varepsilon \varphi$ where $\varphi \in C_0^\infty(\Omega)$ and integrate over Ω:

$$\mu \int_\Omega u_\varepsilon \varepsilon^2 w_\varepsilon \varphi dx + \int_\Omega \sum_i \xi_i^\varepsilon \frac{\partial^2}{\partial x_i^2} (\varepsilon^2 w_\varepsilon \varphi) dx = \int_\Omega f \varepsilon^2 w_\varepsilon \varphi dx . \tag{2.9}$$

Let us look to

$$I_\varepsilon = \int_\Omega \sum_i \xi_i^\varepsilon \left[\frac{\partial^2}{\partial x_i^2} (\varepsilon^2 w_\varepsilon)\varphi + 2 \frac{\partial}{\partial x_i} (\varepsilon^2 w_\varepsilon) \frac{\partial \varphi}{\partial x_i} + \varepsilon^2 w_\varepsilon \frac{\partial^2 \varphi}{\partial x_i^2}\right] dx = a_\varepsilon + b_\varepsilon + c_\varepsilon ,$$

$$b_\varepsilon = 2 \int_\Omega \sum_i \xi_i^\varepsilon \frac{\partial}{\partial x_i} (\varepsilon^2 w_\varepsilon) \frac{\partial \varphi}{\partial x_i} dx \xrightarrow[(\varepsilon \to 0)]{} 2 \int_\Omega \sum_i \xi_i \frac{\partial p}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx ,$$

$$c_\varepsilon = \int_\Omega \sum \xi_i^\varepsilon \varepsilon^2 w_\varepsilon \frac{\partial^2 \varphi}{\partial x_i^2} dx \xrightarrow[(\varepsilon \to 0)]{} \int_\Omega \sum \xi_i p \frac{\partial^2 \varphi}{\partial x_i^2} dx ,$$

$$a_\varepsilon = \int_\Omega \sum_j \frac{\partial^2 u_\varepsilon}{\partial x_j^2} \left(\sum_i a_{ij_\varepsilon} \frac{\partial^2}{\partial x_i^2} (\varepsilon^2 w_\varepsilon)\varphi\right) dx$$

$$= \int_\Omega u_\varepsilon \left[\sum_{i,j} \frac{\partial^2}{\partial x_j^2} \left(a_{ij_\varepsilon} \frac{\partial^2}{\partial x_i^2} (\varepsilon^2 w_\varepsilon)\right)\varphi + 2 \sum_j \frac{\partial}{\partial x_j} \left(\sum_i a_{ij_\varepsilon} \frac{\partial^2}{\partial x_i^2} (\varepsilon^2 w_\varepsilon)\right) \frac{\partial \varphi}{\partial x_j} + \sum_{i,j} a_{ij_\varepsilon} \frac{\partial^2}{\partial x^2} (\varepsilon^2 w_\varepsilon) \frac{\partial^2 \varphi}{\partial x_j^2}\right] .$$

Since $A(\varepsilon^2 w_\varepsilon) = 0$ the first term of the second member is equal to zero and a_ε reduces to

$$a_\varepsilon = \int_\Omega \sum_j \left(\sum a_{ij_\varepsilon} \frac{\partial^2}{\partial x_i^2} (\varepsilon^2 w_\varepsilon)\right) \left[-2 \frac{\partial \varphi}{\partial x_j} \frac{\partial u_\varepsilon}{\partial x_j} - u_\varepsilon \frac{\partial^2 \varphi}{\partial x_j^2}\right] dx$$

and

$$a_\varepsilon \xrightarrow[\varepsilon \to 0]{} \sum_j M\left(\sum_i a_{ij} \left(\frac{\partial^2 p}{\partial x_i^2} - \frac{\partial^2 X}{\partial x_i^2}\right)\right) \int_\Omega -2 \frac{\partial \varphi}{\partial x_j} \frac{\partial u}{\partial x_j} - u \frac{\partial^2 \varphi}{\partial x_j^2} dx$$

$$= \sum_j M\left(\sum_k a_{kj} \left(\frac{\partial^2 p}{\partial x_i^2} - \frac{\partial^2 X}{\partial x_k^2}\right)\right) \int_\Omega \frac{\partial^2 u}{\partial x_j^2} \varphi dx .$$

Going to the limit on (2.9) we get:

(2.10)
$$\mu \int_\Omega uP\varphi dx + 2\int_\Omega \sum \xi_i \frac{\partial P}{\partial x_i}\frac{\partial \varphi}{\partial x_i}dx + \int_\Omega \sum \xi_i P \frac{\partial^2 \varphi}{\partial x_i^2}dx$$
$$+ \sum_j M\left(\sum_k a_{kj}\left(\frac{\partial^2 P}{\partial x_k^2} - \frac{\partial^2 \chi}{\partial x_k^2}\right)\int_\Omega \frac{\partial^2 u}{\partial x_j^2}\varphi dx = (f,\varphi P) \ .$$

On the other hand multiplying $(V)_\varepsilon$ by φP, integrating over Ω and going to the limit we get:

(2.11)
$$\mu \int_\Omega uP\varphi dx + \int_\Omega \sum_i \xi_i \left(P \frac{\partial^2 \varphi}{\partial x_i^2} + 2\frac{\partial P}{\partial x_i}\frac{\partial \varphi}{\partial x_i} + \varphi \frac{\partial^2 P}{\partial x_i^2}\right)dx = (f,\varphi P) \ .$$

From (2.10) and (2.11) it results:

(2.12)
$$\int_\Omega \sum \xi_i \ \frac{\partial^2 P}{\partial x_i^2}dx = \sum_j M\left(\sum_k a_{kj}\left(\frac{\partial^2 P}{\partial x_k^2} - \frac{\partial^2 \chi}{\partial x_k^2}\right)\right)\int_\Omega \frac{\partial^2 u}{\partial x_j^2}\varphi dx \ .$$

Taking $P(x) = \frac{1}{2}x_i^2$:

(2.13)
$$\xi_i = \sum_j M\left(a_{ij} - \sum_k a_{kj}\frac{\partial^2 \chi}{\partial x_k^2}\right)\frac{\partial^2 u}{\partial x_j^2} ;$$

the limit equation is

(2.14)
$$\mu u + \sum_{i,j} M\left(a_{ij} - \sum_k a_{kj}\frac{\partial^2 \chi}{\partial x_k^2}\right)\frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} = f \ .$$

Let us now describe how to deal with the terms of the form $\sum b_{ij_\varepsilon}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial u}{\partial x_j}$ in F_ε:

<u>Theorem 4</u>. Given $(b_i)_{i=1,\dots,n}$, Y-periodic functions, let us define
$$F_\varepsilon(u) = \int_\Omega \left\{(\Delta u)^2 + \mu|Du|^2 + \Delta u \sum b_{i_\varepsilon}\frac{\partial u}{\partial x_i}\right\}dx \ .$$

When ε goes to zero, F_ε converge in $\Gamma^-(w - H_0^2)$ sense (μ is taken large enough) to F_0:
$$F_0(u) = \int_\Omega \left\{(\Delta u)^2 + \mu|Du|^2 - \sum M(\Delta X^i,\Delta X^j)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right\}dx$$

where X^i satisfies
$$\begin{cases}\Delta X^i + b_i - M(b_i) = 0 \\ X^i \text{ is Y-periodic} \ .\end{cases}$$

Considering the Euler equation that means, that for any $f \in H^{-2}$, the solution u_ε of

$$(\mathrm{VI})_\varepsilon \qquad \Delta^2 u_\varepsilon - \mu\Delta u_\varepsilon + \Delta\left(\sum b_{i_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\right) - \sum_i \frac{\partial}{\partial x_i}(b_{i_\varepsilon}\Delta u_\varepsilon) = f$$

converge as $\varepsilon \to 0$ to the solution u of

$$(\mathrm{VI}) \qquad \Delta^2 u - \mu\Delta u + \sum_{i,j} M(\Delta X^i \cdot \Delta X^j)\frac{\partial^2 u}{\partial x_i \partial x_j} = f \quad \text{on } \Omega; \quad u \in H_0^2(\Omega).$$

<u>Proof of Theorem 4</u>. It is clear that for μ large enough the functionals $(F_\varepsilon)_{\varepsilon>0}$ are convex, uniformly coercive on $H_0^2(\Omega)$; the $(u_\varepsilon)_{\varepsilon>0}$ solutions of the corresponding Euler equations stay bounded in $H_0^2(\Omega)$; let $u_\varepsilon \xrightarrow[(\varepsilon \to 0)]{w - H_0^2} u$ and identify u as the solution of the limit equation (VI). The only problem is to compute the limit in weak H^{-1} of the sequence $-\sum_i \frac{\partial}{\partial x_i}(b_{i_\varepsilon}\Delta u_\varepsilon)$; given $\varphi \in C_0^\infty(\Omega)$ let us consider

$$I_\varepsilon = \int_\Omega -\sum \frac{\partial}{\partial x_i}(b_i \Delta u_\varepsilon)\varphi dx = \int_\Omega \sum_i b_{i_\varepsilon}\Delta u_\varepsilon \frac{\partial\varphi}{\partial x_i} dx$$

and introduce X^i solution of (2.15) $\left|\begin{array}{l}\Delta X^i + b_i - M(b_i) = 0 \\ X^i \quad Y\text{-periodic}.\end{array}\right.$

Such X^i exists since $M(b_i - M(b_i)) = 0$. We can now write I_ε in the following way:

$$I_\varepsilon = \int_\Omega \sum_i (M(b_i) - \Delta(\varepsilon^2 X^i_\varepsilon))\Delta u_\varepsilon \frac{\partial\varphi}{\partial x_i} dx ,$$

$$I_\varepsilon = \sum_i M(b_i)\int_\Omega \Delta u_\varepsilon \frac{\partial\varphi}{\partial x_i} dx - \int_\Omega \sum_i \Delta(\varepsilon^2 X^i_\varepsilon)\Delta u_\varepsilon \frac{\partial\varphi}{\partial x_i} dx .$$

The first term of the second member clearly converges to $\sum_i M(b_i)\int_\Omega \Delta u \frac{\partial\varphi}{\partial x_i} dx$. Let us consider

$$J_\varepsilon = -\int_\Omega \sum_i \Delta(\varepsilon^2 X^i_\varepsilon)\Delta u_\varepsilon \frac{\partial\varphi}{\partial x_i} dx$$

and integrate by parts

$$= \sum_i \int_\Omega \varepsilon^2 X^i_\varepsilon\left[\Delta^2 u_\varepsilon \frac{\partial\varphi}{\partial x_i} + 2D(\Delta u_\varepsilon)D\left(\frac{\partial\varphi}{\partial x_i}\right) + \Delta u_\varepsilon \Delta\left(\frac{\partial\varphi}{\partial x_i}\right)\right]dx$$

and using that u_ε satisfies $(\mathrm{VI})_\varepsilon$

$$= \sum_i \int_\Omega \varepsilon^2 X^i_\varepsilon\Bigg[\left(f + \mu\Delta u_\varepsilon + \sum_j \frac{\partial}{\partial x_j}(b_{j_\varepsilon}\Delta u_\varepsilon) - \Delta\left(\sum_j b_{j_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\frac{\partial\varphi}{\partial x_i}$$
$$+ 2D(\Delta u_\varepsilon)D\left(\frac{\partial\varphi}{\partial x_i}\right) + \Delta u_\varepsilon \Delta\left(\frac{\partial\varphi}{\partial x_i}\right)\Bigg]dx .$$

After reduction

$$J_\varepsilon = \sum_i \int_\Omega \varepsilon^2 X^i_\varepsilon \left[\left(f + \mu\Delta u_\varepsilon + \sum_j \frac{\partial}{\partial x_j}(b_{j_\varepsilon})\Delta u_\varepsilon - \sum_j \Delta(b_{j_\varepsilon}) \frac{\partial u_\varepsilon}{\partial x_j} - 2\sum_j D(b_{j_\varepsilon}) D\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right) \frac{\partial\varphi}{\partial x_i}\right.$$

$$\left. + 2D(\Delta u_\varepsilon) D\left(\frac{\partial\varphi}{\partial x_i}\right) + \Delta u_\varepsilon \Delta\left(\frac{\partial\varphi}{\partial x_i}\right)\right] dx .$$

When ε goes to zero, all the terms but one go to zero:

$$\lim_{\varepsilon\to 0} J_\varepsilon = \lim_{\varepsilon\to 0} \sum_i \int_\Omega \varepsilon^2 X^i_\varepsilon \left(-\frac{1}{\varepsilon^2} \sum_j (\Delta(b_j))_\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}\right) \frac{\partial\varphi}{\partial x_i} dx = \int_\Omega \sum_{i,j} M(X^i\Delta(b_j)) \frac{\partial u}{\partial x_j} \frac{\partial\varphi}{\partial x_i} dx .$$

Finally,

$$I_\varepsilon \xrightarrow[(\varepsilon \to 0)]{} \sum_i M(b_i) \int_\Omega \Delta u \frac{\partial\varphi}{\partial x_i} dx + \sum_{i,j} M(X^i\Delta(b_j)) \int_\Omega \frac{\partial u}{\partial x_j} \frac{\partial\varphi}{\partial x_i} dx ;$$

that is to say:

$$-\sum_i \frac{\partial}{\partial x_i}(b_{i_\varepsilon}\Delta u_\varepsilon) \xrightarrow[(\varepsilon \to 0)]{w - H^{-1}} -\sum M(b_i) \frac{\partial}{\partial x_i}\Delta u + \sum_{i,j} M(X^i\Delta(b_j)) \frac{\partial^2 u}{\partial x_i \partial x_j} .$$

Since

$$\Delta\left(\sum_i b_{i_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\right) \xrightarrow[\varepsilon \to 0]{w - H^{-2}} \Delta\left(\sum_i M(b_i) \frac{\partial u}{\partial x_i}\right) = \sum_i M(b_i) \Delta\left(\frac{\partial u}{\partial x_i}\right)$$

the limit equation is:

(VI) $$\Delta^2 u - \mu\Delta u - \sum M(X^i\Delta b_j) \frac{\partial^2 u}{\partial x_i \partial x_j} = f \quad \text{on } \Omega; \quad u \in H^2_0(\Omega) .$$

We remark that

$$M(X^i\Delta b_j) = -\frac{1}{|Y|} \int_Y \Delta X^i \Delta X^j dx$$

and that u minimizes $\frac{1}{2} F_0 - (f,\cdot)$: with

$$F_0(u) = \int_\Omega \left\{(\Delta u)^2 + \mu|Du|^2 - \sum M(\Delta X^i\Delta X^j) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\right\} dx .$$

Let us now consider the last type of higher order terms that appears in (2.6):

<u>Theorem 5</u>. Let us consider the sequence of functionals $(F_\varepsilon)_{\varepsilon>0}$

$$F_\varepsilon(u) = \int_\Omega \left\{(\Delta u)^2 + \mu u^2 + \sum_i a_{i_\varepsilon} u \frac{\partial^2 u}{\partial x_i^2}\right\} dx$$

where the a_i are Y-periodic and μ is taken large enough in order for the $(F_\varepsilon)_{\varepsilon>0}$ to be uniformly coercive on $H_0^2(\Omega)$. When ε goes to zero, F_ε converge in $\Gamma^-(w - H_0^2)$ sense to F_0:

$$F_0(u) = \int_\Omega [(\Delta u)^2 + |Du|^2 - \frac{1}{2} M((\Delta X)^2)u^2]dx$$

where

$$\Delta^2 X + \sum \frac{\partial^2 a_i}{\partial x_i^2} = 0; \quad X \text{ is Y-periodic .}$$

Considering the Euler equation that means that u solution of:

$$\text{(VII)}_\varepsilon \qquad \Delta^2 u_\varepsilon + \mu u_\varepsilon + \sum \frac{\partial^2}{\partial x_i^2} (a_{i_\varepsilon} u_\varepsilon) + \sum a_{i_\varepsilon} \frac{\partial^2 u_\varepsilon}{\partial x_i^2} = f$$

converge in weak-H_0^2 to u solution of

$$\text{(VII)} \qquad \Delta^2 u + \mu u - M((\Delta X)^2)u = f .$$

Proof of Theorem 5. The proof is just slightly different from the one of the model problem $(IV)_\varepsilon$: The only problem is to compute the limit of the sequence $\left(\sum a_{i_\varepsilon} \frac{\partial^2 u_\varepsilon}{\partial x_i^2}\right)_{\varepsilon>0}$. Let us introduce X solution of

$$\text{(2.16)} \qquad \left| \begin{array}{l} \Delta^2 X + \sum \dfrac{\partial^2 a_i}{\partial x_i^2} = 0 \\ X \text{ is Y-periodic .} \end{array} \right.$$

From (2.16), (2.17)

$$\Delta^2(\varepsilon^2 X_\varepsilon) + \sum \frac{\partial^2}{\partial x_i^2} (a_{i_\varepsilon}) = 0 ;$$

multiplying (2.17) by φu_ε, $\varphi \in C_0^\infty(\Omega)$:

$$\text{(2.18)} \qquad \int_\Omega \Delta(\varepsilon^2 X_\varepsilon)\Delta(\varphi u_\varepsilon)dx + \int_\Omega \sum_i a_{i_\varepsilon} \frac{\partial^2}{\partial x_i^2} (\varphi u_\varepsilon)dx = 0 ,$$

$$\text{(2.19)} \qquad \int_\Omega (\Delta X)_\varepsilon (\Delta\varphi u_\varepsilon + 2D\varphi Du_\varepsilon)dx + \int_\Omega \Delta(\varepsilon^2 X_\varepsilon)\varphi\Delta u_\varepsilon dx$$
$$+ \int_\Omega \sum_i a_{i_\varepsilon}\left(\frac{\partial^2\varphi}{\partial x_i^2} u_\varepsilon + \frac{2\partial\varphi}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_i} + \frac{\varphi\partial^2 u_\varepsilon}{\partial x_i^2}\right)dx = 0$$

which implies, since $(\Delta X)_\varepsilon \xrightarrow{w - L^2} 0$,

$$(2.20) \qquad \lim_{\varepsilon\to 0}\left\{\int_\Omega \Delta(\varepsilon^2 X_\varepsilon)\varphi\Delta u_\varepsilon dx + \int_\Omega \varphi\left(\sum_i a_{i_\varepsilon}\frac{\partial^2 u_\varepsilon}{\partial x_i^2}dx\right)\right\} = \sum_i M(a_i)\int_\Omega\left\{\frac{\partial^2\varphi}{\partial x_i^2}u + \frac{2\partial\varphi}{\partial x_i}\frac{\partial u}{\partial x_i}\right\}dx$$

$$= -\sum_i M(a_i)\int_\Omega \frac{\partial^2 u}{\partial x_i^2}\varphi dx .$$

Let us consider

$$J_\varepsilon = \int_\Omega \Delta(\varepsilon^2 X_\varepsilon)\varphi\Delta u_\varepsilon dx ,$$

integrate by parts, and use that u_ε satisfies $(VII)_\varepsilon$:

$$J_\varepsilon = \int_\Omega \varepsilon^2 X_\varepsilon\left[\Delta\varphi\Delta u_\varepsilon + 2D\varphi D(\Delta u_\varepsilon) + \varphi\left(f - \mu u_\varepsilon - 2\sum_i a_{i_\varepsilon}\frac{\partial^2 u_\varepsilon}{\partial x_i^2} - 2\sum_i \frac{\partial a_{i_\varepsilon}}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_i} - \sum_i \frac{\partial^2}{\partial x_i^2}(a_{i_\varepsilon})u_\varepsilon\right)\right]dx .$$

When ε goes to zero

$$(2.21) \qquad J_\varepsilon \xrightarrow[(\varepsilon\to 0)]{} -\sum_i M\left(X\frac{\partial^2 a_i}{\partial x_i^2}\right)\int_\Omega \varphi u dx .$$

From (2.20) and (2.21) it follows

$$(2.22) \qquad \int_\Omega \varphi\left(\sum_i a_{i_\varepsilon}\frac{\partial^2 u_\varepsilon}{\partial x_i^2}\right)dx \xrightarrow[(\varepsilon\to 0)]{} \sum_i M\left(X\frac{\partial^2 a_i}{\partial x_i^2}\right)\int_\Omega \varphi u dx - \sum_i M(a_i)\int_\Omega \frac{\partial^2 u}{\partial x_i^2}\varphi dx$$

which means:

$$\sum_i a_{i_\varepsilon}\frac{\partial^2 u_\varepsilon}{\partial x_i^2} \xrightarrow{w - L^2} -\sum_i M(a_i)\frac{\partial^2 u}{\partial x_i^2} + \sum_i M\left(X\frac{\partial^2 a_i}{\partial x_i^2}\right)u .$$

The limit equation is:

$$\Delta^2 u + \mu u + \sum M(a_i)\frac{\partial^2 u}{\partial x_i^2} - \sum M(a_i)\frac{\partial^2 u}{\partial x_i^2} + \sum_i M\left(X\frac{\partial^2 a_i}{\partial x_i^2}\right)u = f ,$$

$$(VII) \qquad \Delta^2 u + \sum_i M\left(X\frac{\partial^2 a_i}{\partial x_i^2}\right)u = f .$$

From (2.16),

$$M\left(\sum_i X\frac{\partial^2 a_i}{\partial x_i^2}\right) = -M((\Delta X)^2) .$$

Remark 7. From Theorem 5 one can refind easily the result of the model problem where

$$F_\varepsilon(u) = \int_\Omega \frac{1}{2}\mu u^2 + \frac{1}{2}(\Delta u)^2 - 2(\Delta X)_\varepsilon u\Delta u - 2(\Delta X)_\varepsilon |Du|^2 dx .$$

It is clear that the contribution of the last term, in $\Gamma^-(w - H_0^2)$ convergence, is zero; so, we apply Theorem 5, with $a_i = \Delta X$ and get the result of Proposition 2, remarking that $M(WX) = -M((\Delta X)^2)$.

Let us give finally the following example which is relevant of the same type of technics:

Theorem 6. Let $(u_\varepsilon)_{\varepsilon>0}$ be the solutions of the following equations

$$(\mathrm{VIII})_\varepsilon \qquad \mu u_\varepsilon + \Delta^2 u_\varepsilon + W_{0,\varepsilon}\Delta u_\varepsilon + \frac{1}{\varepsilon} W_{1,\varepsilon} \operatorname{div} u_\varepsilon + \frac{1}{\varepsilon^2} W_{2,\varepsilon} u_\varepsilon = f \text{ on } \Omega; \qquad u_\varepsilon \in H_0^2(\Omega)$$

(we take μ large enough) where the W_i $(i = 0,1,2)$ are Y-periodic functions with zero mean value.

When ε goes to zero, u_ε converge weakly in $H_0^2(\Omega)$ to u solution of

$$(\mathrm{VIII}) \qquad \mu u + \Delta^2 u + M[W_2(X_2 - \operatorname{div} X_1 + \Delta X_0)]u = f \text{ on } \Omega; \quad u \in H_0^2(\Omega)$$

where X_i is a solution of (2.23) $\begin{cases} \Delta^2 X_i + W_i = 0 \\ X_i \text{ Y-periodic .} \end{cases}$

Proof of Theorem 6. Let us consider

$$I_0^\varepsilon = \int_\Omega W_{0,\varepsilon}\Delta u_\varepsilon \varphi dx, \quad I_1^\varepsilon = \int_\Omega \frac{1}{\varepsilon} W_{1,\varepsilon} \operatorname{div} u_\varepsilon \varphi dx, \quad I_2^\varepsilon = \int_\Omega \frac{1}{\varepsilon^2} W_{2,\varepsilon} u_\varepsilon \varphi dx$$

where $\varphi \in C_0^\infty(\Omega)$ and compute their respective limits when $\varepsilon \to 0$. Just like for the model problem, one can prove

$$(2.24) \qquad I_2^\varepsilon \xrightarrow[(\varepsilon \to 0)]{} M(W_2 X_2) \int_\Omega u\varphi dx ;$$

let us consider now I_0^ε

$$I_0^\varepsilon = \int_\Omega W_{0,\varepsilon}\Delta u_\varepsilon \varphi dx = -\varepsilon^4 \int_\Omega \Delta X_{0,\varepsilon}\Delta u_\varepsilon \varphi dx ,$$

$$I_0^\varepsilon = -\varepsilon^4 \int_\Omega \Delta X_{0,\varepsilon}[\Delta^2 u_\varepsilon \varphi + 2D(\Delta u_\varepsilon)D\varphi + \Delta u_\varepsilon \Delta\varphi]dx$$

$$= -\varepsilon^2 \int_\Omega (\Delta X_0)_\varepsilon [f - \mu u_\varepsilon - W_{0,\varepsilon}\Delta u_\varepsilon - \frac{1}{\varepsilon} W_{1,\varepsilon} \text{div } u_\varepsilon - \frac{1}{\varepsilon^2} W_{2,\varepsilon} u_\varepsilon]\varphi dx$$

$$-\varepsilon^2 \int_\Omega (\Delta X_0)_\varepsilon [2D(\Delta u_\varepsilon)D\varphi + \Delta u_\varepsilon \Delta\varphi]dx \ .$$

When ε goes to zero we see that

(2.25)
$$I_0^\varepsilon \to M(\Delta X_0 W_2) \int_\Omega u\varphi dx \ .$$

Let us now look to the last term I_1^ε:

$$I_1^\varepsilon = \int_\Omega \frac{1}{\varepsilon} W_{1,\varepsilon} \text{div } u_\varepsilon \varphi dx = -\varepsilon^3 \int_\Omega \Delta^2(X_{1,\varepsilon}) \text{div } u_\varepsilon \varphi dx \ ,$$

$$I_1^\varepsilon = \varepsilon^3 \int_\Omega u_\varepsilon [\Delta^2(X_{1,\varepsilon})\text{div}\varphi + \Delta^2(\text{div } X_{1,\varepsilon})\varphi]dx \ ,$$

$$I_1^\varepsilon = \varepsilon^3 \int_\Omega \Delta(X_{1,\varepsilon})\Delta(u_\varepsilon \text{div}\varphi)dx + \varepsilon^3 \int_\Omega \Delta(\text{div } X_{1,\varepsilon})\Delta(u_\varepsilon\varphi)dx = J_1^\varepsilon + H_1^\varepsilon \ ,$$

$$J_1^\varepsilon = \varepsilon \int_\Omega (\Delta X_1)_\varepsilon (\Delta u_\varepsilon \text{div}\varphi + 2Du_\varepsilon D(\text{div}\varphi) + u_\varepsilon \text{div}(\Delta\varphi))dx \xrightarrow[(\varepsilon \to 0)]{} 0 \ ,$$

$$H_1^\varepsilon = \varepsilon^2 \int_\Omega (\text{div } X_1)_\varepsilon \Big[\Delta^2 u_\varepsilon \varphi + \Delta u_\varepsilon \Delta\varphi + 2D(\Delta u_\varepsilon)D\varphi + \Delta u_\varepsilon \Delta\varphi + u_\varepsilon \Delta^2\varphi + 2Du_\varepsilon D(\Delta\varphi)$$

$$+ 2D(\Delta u_\varepsilon)D\varphi + 2Du_\varepsilon D(\Delta\varphi) + 4 \sum_i D\left(\frac{\partial u_\varepsilon}{\partial x_i}\right) D\left(\frac{\partial\varphi}{\partial x_i}\right)\Big]dx \ .$$

Using that u_ε satisfies $(\text{VIII})_\varepsilon$ we get that

(2.26)
$$\lim_{\varepsilon\to 0} I_1^\varepsilon = \lim_{\varepsilon\to 0} H_1^\varepsilon = \lim_{\varepsilon\to 0} \int_\Omega \varepsilon^2 (\text{div } X_1)_\varepsilon \Big[-\frac{1}{\varepsilon^2} W_{2,\varepsilon} u_\varepsilon\Big]\varphi dx = -M(\text{div } X_1 W_2) \int_\Omega u\varphi dx \ .$$

From (2.24), (2.25) and (2.26) it follows that

$$W_{0,\varepsilon}\Delta u_\varepsilon + \frac{1}{\varepsilon} W_{1,\varepsilon} \text{div } u_\varepsilon + \frac{1}{\varepsilon^2} W_{2,\varepsilon} u_\varepsilon \xrightarrow[\varepsilon \to 0]{} M[W_2(X_2 - \text{div } X_1 + \Delta X_0)]u_0$$

and u satisfies (VIII).

2.3. Energetic interpretation and compactness results for higher order functionals.

We are going to see that the stability results we got in the preceding paragraph are relevant of general compactness theorems for the family of functionals

$$F_h(u) = \int_\Omega f_h(x, u, Du, D^2u)dx$$

where f_h is Caratheodory, convex continuous in (u, Du, D^2u) and satisfies

$$\lambda_0 |z|^p \leq f_h(x,\xi,y,z) \leq \Lambda_0(1 + |\xi|^p + |y|^p + |z|^p) \qquad (p > 1) .$$

Theorem 7. Let

$$f_h : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{2n} \to \mathbb{R}^+$$
$$(x,\xi,y,z) \mapsto f_h(x,\xi,y,z)$$

be a sequence of convex integrands satisfying

$$(C) \qquad \begin{cases} x \mapsto f_h(x,\xi,y,z) \text{ is measurable} \\ (\xi,y,z) \mapsto f_h(x,\xi,y,z) \text{ is convex continuous} \\ \lambda_0(|\xi|^p + |y|^p + |z|^p) \leq f_h(x,\xi,y,z) \leq \Lambda_0(1 + |\xi|^p + |y|^p + |z|^0) \qquad (p > 1) . \end{cases}$$

With such an integrand f_h we associate the functional F_h:

$$\forall \Omega \text{ open bounded set in } \mathbb{R}^n, \ \forall u \in W^{2,p}_{loc}(\mathbb{R}^n) \quad F_h(u,\Omega) = \int_\Omega f_h(x,u(x),Du(x),D^2u(x))dx .$$

There exists a subsequence $(h(k))_{k\in\mathbb{N}}$ and an integrand f, still satisfying (C), such that:

$$\forall \Omega \text{ open bounded set in } \mathbb{R}^n, \ \forall u \in W^{2,p}(\Omega) \quad F(u,\Omega) = \Gamma^-(w - W^{2,p}(\Omega)) \lim_{k\to+\infty} F_{h_k}(u,\Omega)$$

where, $F(u,\Omega) = \int_\Omega f(x,u,Du,D^2u)dx$ is the functional associated with f.

The conclusion still holds with $W^{2,p}_0(\Omega)$ instead of $W^{2,p}(\Omega)$.

Proof of Theorem 7. The proof is very similar to the proof of Theorem A ([1]); so, we shall develop essentially the parts where the introduction of higher order terms bring some modifications.

Step 1.

Let $\mathcal{B}_n$ be a denumerable rich family of open regular sets in $\mathbb{R}^n$ (by regular we mean that the boundary is of zero Lebesgue measure).

By the classical abstract compactness theorem of Kuratowski, and using a diagonalization lemma, we can extract a subsequence $(h(k))_{k\in\mathbb{N}}$ such that:

$$(2.27) \qquad \forall \Omega \in \mathcal{B}_n \ \forall u \in W^{2,p}_{loc}(\mathbb{R}^n) \quad \Gamma^-(w - W^{2,p}(\Omega)) \lim_{k\to+\infty} F_{h_k}(u,\Omega)$$

exists. From now on we shall write F_k instead of F_{h_k}: (2.27) means that

$$\forall \Omega \in \mathcal{B}_n \quad \forall u \in W^{2,p}_{loc}(\mathbb{R}^n) \quad F'(u,\Omega) = F''(u,\Omega)$$

where

$$\begin{cases} F'(u,\Omega) = \Gamma^-(w - W^{2,p})\lim\inf F_k(u,\Omega) \\ F''(u,\Omega) = \Gamma^-(w - W^{2,p})\lim\sup F_k(u,\Omega) \ . \end{cases}$$

Since F' and F'' are increasing functions of Ω and $\mathcal{B}_n$ is rich, we get:

$$\forall u \in W^{2,p}_{loc}(\mathbb{R}^n) \ \forall \Omega \text{ open bounded set in } \mathbb{R}^n \quad H(u,\Omega) = \sup_{w \subset\subset \Omega} F'(u,w) = \sup_{w \subset\subset \Omega} F''(u,w) \ .$$

<u>Step 2.</u>

$\Omega \to H(u,\Omega)$ is the restriction to open bounded sets of a regular Borel measure. Since $H(u,\cdot)$ is an increasing, inner regular function of Ω we have just to prove that it is additive and subadditive.

Clearly $\Omega \mapsto F'(u,\Omega)$ is superadditive; the conclusion will follow from: $\Omega \mapsto F''(u,\Omega)$ is subadditive on $\mathcal{B}_n$:

$$\forall \Omega_1,\Omega_2 \in \mathcal{B}_n \quad \forall u \in W^{2,p}_{loc}(\mathbb{R}^n) \quad F''(u,\Omega_1 \cup \Omega_2) \le F''(u,\Omega_1) + F''(u,\Omega_2) \ .$$

Noticing that

$$\Omega_1 \cup \Omega_2 = \Omega_1 \cup (\Omega_2 \setminus \Omega_1) \supset \Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1})$$

and $\text{meas}(\Omega_1 \cup \Omega_2) \setminus (\Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1}))$ is equal to zero, we reduce proving that: $\forall w_1,w_2,\Omega$ open bounded sets such that

$$\Omega \supset w_1 \cup w_2, \quad \text{meas}(\Omega \setminus (w_1 \cup w_2)) = 0$$

we have:

$$\forall u \in W^{2,p}_{loc}(\mathbb{R}^n) \quad F''(u,\Omega) \le F''(u,w_1) + F''(u,w_2) \ .$$

Let us introduce

$$\Delta = \Omega \setminus (w_1 \cup w_2)$$

(by hypothesis $\text{meas}(\Delta) = 0$) and

$$I_r(\Delta) = \{x \in \mathbb{R}^n / \text{dist}(x,\Delta) \le r\} \ .$$

By Urisohn's lemma, there exist φ regular (here we need $\varphi \in C^2$) such that:

$$\varphi = 1 \text{ on } I_r(\Delta) \text{ and } \varphi = 0 \text{ outside of } I_{2r}(\Delta) \ .$$

(From φ Lipschitz, one can get $\varphi \in C^2$ by regularization by convolution, noticing that the thickness of $I_{2r} \setminus I_r$ is strictly positive.)

By definition there exist $u_k^i \xrightarrow{w - W^{2,p}(W_i)} u$ such that

(2.28) $$F''(u,W_i) = \limsup F_k(u_k^i,W_i) \qquad (i = 1,2) .$$

Let us define:

$$v_k = \begin{cases} (1-\varphi)u_k^1 + \varphi u & \text{on } W_1 \\ u & \text{on } I_r(A) \\ (1-\varphi)u_k^2 + \varphi u & \text{on } W_2 . \end{cases}$$

Clearly $v_k \to u$ in $L^p(\Omega)$; let us prove that

$$\sup_k F_k(v_k,\Omega) < +\infty ;$$

from the uniform coerciveness of the F_k and the definition of F'' that will imply: $v_k \xrightarrow{w - W^{2,p}(\Omega)} u$ and

$$F''(u,\Omega) \le \limsup_{k\to+\infty} F_k(v_k,\Omega) .$$

In fact let us prove this inequality in tu, $0 < t < 1$:

$$F_k(tv_k,\Omega) = \sum_{i=1}^{2} \int_{W_i} f_k\Big[x, t(1-\varphi)u_k^i + t\varphi u, t(1-\varphi)Du_k^i + t\varphi Du + tD\varphi(u-u_k^i), t(1-\varphi)D^2u_k^i$$
$$+ t\varphi D^2u + tD^2\varphi(u-u_k^i) + t\left[\frac{\partial}{\partial x_\ell}(u-u_k^i)\frac{\partial\varphi}{\partial x_p} + \frac{\partial\varphi}{\partial x_\ell}\frac{\partial}{\partial x_p}(u-u_k^i)\right]_{p,\ell}\Big)dx .$$

By convexity of f_k we get

$$F_k(tv_k,\Omega) \le \sum_{i=1}^{2}\int_{W_i} f_k(x,u_k^i,Du_k^i,D^2u_k^i)dx + \int_{I_{2r}(A)} f_k(x,u,Du,D^2u)dx$$
$$+ (1-t)\sum_{i=1}^{2}\int_{W_i} f_k\Big[x,0,\frac{t}{1-t}D\varphi(u-u_k^i), \frac{t}{1-t}\Big[D^2\varphi(u-u_k^i)$$
$$+ \frac{\partial}{\partial x_\ell}(u-u_k^i)\frac{\partial\varphi}{\partial x_p} + \frac{\partial\varphi}{\partial x_\ell}\frac{\partial}{\partial x_p}(u-u_k^i)\Big]\Big]dx .$$

Going to the limit sup, as $k \to +\infty$, and using (2.28) and (c) we get:

$$\limsup_{k\to+\infty} F_k(tv_k,\Omega) \le F''(u,W_1) + F''(u,W_2) + \Lambda_0\int_{I_{2r}(A)} (1 + |u|^p + |Du|^p + |D^2u|^p)dx$$
$$+ \Lambda_0(1-t)\sum_{i=1}^{2}\limsup_{k\to+\infty}\int_{W_i} 1 + \left|\frac{t}{1-t}D\varphi(u-u_k^i)\right|^p$$
$$+ \left(\frac{t}{1-t}\right)^p\left|D^2\varphi(u-u_k^i) + \frac{\partial}{\partial x_\ell}(u-u_k^i)\frac{\partial\varphi}{\partial x_p} + \frac{\partial}{\partial x_p}(u-u_k^i)\frac{\partial\varphi}{\partial x_\ell}\right|^p dx .$$

Since u_k^i converge weakly to u in $W^{2,p}(w_i)$, it converges strongly in $W^{1,p}(w_i)$ which implies:

$$\limsup_{k\to+\infty} F_k(tv_k,\Omega) \le F''(u,w_1) + F''(u,w_2) + \Lambda_0 \int_{I_{2r(\Delta)}} (1+|u|^p+|Du|^p+|D^2u|^p)dx$$

$$+ (1-t)\Lambda_0 \int_\Omega dx \ .$$

Therefore,

$$\limsup_{k\to+\infty} F_k(tv_k,\Omega) < +\infty, \quad tv_k \xrightarrow{w - W^{2,p}(\Omega)} tu$$

and

$$F''(tu,\Omega) \le \limsup_{k\to+\infty} F_k(tv_k,\Omega) \le F''(u,w_1) + F''(u,w_2)$$

$$+ \Lambda_0 \int_{I_{2r(\Delta)}} (1 + |u|^p + |Du|^p + |D^2u|^p)dx + (1-t)\Lambda_0 \int_\Omega dx \ .$$

Making r go to zero, t go to one, and using the lower semicontinuity of $u \mapsto F''(u,\Omega)$ we finally get:

$$F''(u,\Omega) \le F''(u,w_1) + F''(u,w_2) \ .$$

Step 3.

$$H(u,\Omega) = \Gamma^-(w - W^{2,p}(\Omega))\lim F_{h_k}(u,\Omega) = F(u,\Omega) \ .$$

We have just to prove that

$$F''(u,\Omega) \le H(u,\Omega) \le F'(u,\Omega) \ .$$

The right inequality is evident; the left one is a straight-forward consequence of the step 2.

Given $\varepsilon > 0$, let $w_\varepsilon \subset w$, w_ε regular such that $\mathrm{meas}(\Omega\setminus w_\varepsilon) < \varepsilon$; by the preceding argument

$$F''(u,\Omega) \le F''(u,w_\varepsilon) + F''(u,\Omega\setminus\overline{w_\varepsilon})$$

$$\le H(u,\Omega) + \Lambda_0 \int_{\Omega\setminus\overline{w_\varepsilon}} (1 + |u|^p + |Du|^p + |D^2u|^p)dx \ .$$

Making ε go to zero, we get $F''(u,\Omega) \le H(u,\Omega)$.

Step 4.

Let us prove that there exists $f \in (C)$ such that

$$\forall u \in W^{2,p}_{loc}(\mathbb{R}^n) \quad \forall \Omega \text{ bounded open set in } \mathbb{R}^n, \quad F(u,\Omega) = \int_\Omega f(x,u(x),Du(x),D^2u(x))dx .$$

From (C), $\forall u \in W^{2,p}_{loc}(\mathbb{R}^n)$ $\forall \Omega$ bounded open set in $\mathbb{R}^n$

$$0 \le F(u,\Omega) \le \Lambda_0 \int_\Omega (1 + |u|^p + |Du|^p + |D^2u|^p)dx .$$

It follows that the measure $\Omega \to F(u,\Omega)$ is absolutely continuous with respect to the measure of density $(1 + |u|^p + |Du|^p + |D^2u|^p)dx$. By Radon-Nikodym theorem, for any $u \in W^{2,p}_{loc}(\mathbb{R}^n)$ there exists a locally integrable function f_u such that:

$$\forall \Omega \text{ open bounded set in } \mathbb{R}^n, \quad F(u,\Omega) = \int_\Omega f_u(x)dx .$$

Given x_0 in $\mathbb{R}^n$, let us consider the function

$$X_{x_0,\xi,y,z} : x \mapsto \xi + \sum_i (x_i - x_{0_i})y_i + \frac{1}{2}\sum_{i,j} z_{ij}(x_i - x_{0_i})(x_j - x_{0_j}) .$$

There exists a function $f_{x_0,\xi,y,z}$ such that

$$\forall \Omega \text{ open bounded set in } \mathbb{R}^n, \quad F(X_{x_0,\xi,y,z},\Omega) = \int_\Omega f_{x_0,\xi,y,z}(x)dx .$$

Let us prove that

$$\forall u \in W^{2,p}_{loc}(\mathbb{R}^n), \ \forall \Omega \text{ open bounded set}, \quad F(u,\Omega) = \int_\Omega f_{x,u(x),Du(x),D^2u(x)}(x)dx$$

that is to say

$$f_u(x) = f_{x,u(x),Du(x),D^2u(x)}(x) = f(x,u(x),Du(x),D^2u(x)) .$$

Clearly, by a continuity argument, we can reduce taking $u \in C^2(\mathbb{R}^n)$.

Let us take x_0 a Lebesgue point of f_u and let us design by $B(x_0,\rho)$ the open ball in $\mathbb{R}^n$ of center x_0 and radius $\rho > 0$. From (C)

$$\forall v \in W^{2,p}_{loc}(\mathbb{R}^n) \quad \forall \rho > 0, \quad 0 \le F(v,B(x_0,\rho)) \le \Lambda_0(1 + \|v\|^p_{W^{2,p}(B(x_0,\rho))}) .$$

Since $v \mapsto F(v,\Omega)$ is convex it follows classically that there exist a constant $C > 0$ such that

$$\forall u,v \in W^{2,p}_{loc}(\mathbb{R}^n) \quad \forall \rho > 0 \quad |F(u,B(x_0,\rho)) - F(v,B(x_0,\rho))|$$

$$\le C\|u - v\|_{W^{2,p}(B(x_0,\rho))} (1 + \|u\|^p_{W^{2,p}(B(x_0,\rho))} + \|v\|^p_{W^{2,p}(B(x_0,\rho))}) ,$$

$$\forall u,v \in W^{2,p}_{loc}(\mathbb{R}^n) \quad \forall 0 < \rho < 1 \quad |F(u,B(x_0,\rho)) - F(v,B(x_0,\rho))|$$

$$\le C\|u - v\|_{W^{2,p}(B(x_0,\rho))} (1 + \|u\|^p_{W^{2,p}(B(x_0,1))} + \|v\|^p_{W^{2,p}(B(x_0,1))}) .$$

Let us apply this inequality with u and $v = X_{x_0,u(x_0),Du(x_0),D^2u(x_0)}$

$$\forall \rho \; 0 < \rho < 1 \quad \left|\int_{B(x_0,\rho)} f_u(x) - f_{x_0,u(x_0),Du(x_0),D^2u(x_0)}(x)dx\right| \le C\left|\int_{B(x_0,\rho)} |u(x) - X(x)|^p\right.$$

$$\left. + |Du(x) - DX(x)|^p + |D^2u(x) - D^2X(x)|^p dx\right|^{1/p} \cdot (1 + \|u\|^p_{W^{2,p}(B(x_0,1))} + \|DX\|^p_{W^{2,p}(B(x_0,1))}) .$$

We divide this inequality by $|B(x_0,\rho)|$ and make ρ go to zero. Since u is in $C^2(\Omega)$ and $X(x_0) = u(x_0)$, $DX(x_0) = Du(x_0)$, $D^2X(x_0) = D^2u(x_0)$ we get

$$\lim_{\rho\to 0} \frac{1}{|B(x_0,\rho)|} \left|\int_{B(x_0,\rho)} (f_u(x) - f_{x_0,u(x_0),Du(x_0),D^2u(x_0)}(x))dx\right| = 0 .$$

Since x_0 is a Lebesgue point of f_u, we finally obtain

$$f_u(x_0) = f_{x_0,u(x_0),Du(x_0),D^2u(x_0)}(x_0) .$$

<u>Step 5.</u>

$$F(u,\Omega) = \Gamma^-(w - W^{2,p}_0(\Omega))\lim F_{h_k}(u,\Omega) .$$

We adapt the classical proof which consists multiplying u by a function φ regular, equal to zero near the boundary, using the same type of argument as in Step 2.

<u>Remark 8.</u> It is clear that this demonstration can be straightforward extended to functionals of any order. We restricted ourselves to functionals of order two only to simplify the notations.

Let us now give the proof of the homogenization formula for the functionals

$$F_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, D^2u)dx$$

where f is periodic in x, convex, continuous, coercive with respect to D^2u. Since the argument is close to the one developed in [3] we shall just sketch the proof and develop the modifications that the introduction of higher order terms brings:

Theorem 8. Let

$$F_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, D^2u(x))dx$$

where f is Y-periodic in x, convex continuous in $z \in \mathbb{R}^{2n}$ and satisfies:

$$\lambda_0 |z|^p \leq f(x,z) \leq \Lambda_0(1 + |z|^p) \qquad (p > 1) .$$

When ε goes to zero F_ε converges in $\Gamma^-(w - W_0^{2,p}(\Omega))$ sense to F_0:

$$F_0(u) = \int_\Omega f_0(D^2u(x))dx$$

with

$$f_0(z) = \underset{\substack{u \in W^{2,p} \\ u \text{ Y-periodic}}}{\text{Min}} \frac{1}{|Y|} \int_Y f(x, D^2u(x) + z)dx .$$

Proof of Theorem 8. Step 1.

From the compactness Theorem 7, we can extract a subsequence $(\varepsilon_k)_{k \in \mathbb{N}}$ and find an integrand f_0 such that for every Ω open bounded set in $\mathbb{R}^n$ and u in $W^{2,p}_{loc}(\mathbb{R}^n)$,

$$\Gamma^-(w - W^{2,p}(\Omega)) \lim_{k \to +\infty} F_{\varepsilon_k}(u,\Omega) = F_0(u,\Omega)$$

with

$$F_0(u,\Omega) = \int_\Omega f_0(x, D^2u(x))dx$$

(it is clear that f_0 does not depend on u and Du, since $0 \leq f_0(x,\xi,y,z) \leq \Lambda_0(1 + |z|^p)$ and $(\xi,y) \mapsto f_0(x,\xi,y,z)$ is convex).

The convergence result will follow from the identification of f_0.

Step 2.

f_0 is independent of x.

The idea is the same as the one developed in Proposition 1; let us see how to extend it simply to the most general case where

$$F_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, u, Du, \ldots, D^\nu u)dx$$

(for simplicity of notations let us take $\nu = 2$).

From the proof of Theorem 7

$$F_0(X_{x_1,\xi,y,z},\ ) = \int_\Omega f(x_1,x,\xi,y,z)dx$$

where

$$X_{x_1,\xi,y,z}(x) = \xi + \langle y, x - x_1\rangle + \frac{1}{2}\langle z, x - x_1, x - x_1\rangle$$

and

$$F_0(u,\Omega) = \int_\Omega f_0(x,u,Du,D^2u)dx \quad \text{with} \quad f_0(x,\xi,y,z) = f(x,x,\xi,y,z) \ .$$

Let x_0, x_1, x_2 be fixed in $\mathbb{R}^n$ and n_k^i such that $n_k^i\varepsilon_k \le x_0^i \le (n_k^i + 1)\varepsilon_k$. Let $u_{\varepsilon_k} \xrightarrow{w - W^{2,p}(\Omega)} X_{x_1,\xi,y,z}$ and

$$F_0(X_{x_1,\xi,y,z},\Omega) = \lim_{k\to+\infty} F_{\varepsilon_k}(u_{\varepsilon_k},\Omega) \quad \text{with} \quad \Omega = Y_\varepsilon + x_2 \ .$$

$$F_{\varepsilon_k}(u_{\varepsilon_k},\Omega) = \int_{\Omega + n_k^i\varepsilon_k} f\left(\frac{x}{\varepsilon_k}, u_{\varepsilon_k}(x - n_k^i\varepsilon_k), Du_{\varepsilon_k}(x - n_k^i\varepsilon_k), D^2u_{\varepsilon_k}(x - n_k^i\varepsilon_k)\right)dx \ .$$

Let us extend $u_{\varepsilon_k}(\cdot - n_k^i\varepsilon_k)$ to $\Omega + x_0$ and v_k be the extension such that

$$|F_{\varepsilon_k}(u_{\varepsilon_k},\Omega) - F_{\varepsilon_k}(v_k,\Omega + x_0)| \xrightarrow[k\to+\infty]{} 0 \ .$$

Since v_k converges to $X_{x_1,\xi,y,z}(\cdot - x_0)$ in $w - W^{2,p}(\Omega + x_0)$,

$$\begin{aligned} F_0(X_{x_1,\xi,y,z}(\cdot - x_0), Y_\varepsilon + x_0 + x_2) &\le \liminf_{k\to+\infty} F_{\varepsilon_k}(v_k, Y_\varepsilon + x_2 + x_0) \\ &\le \liminf_{k\to+\infty} F_{\varepsilon_k}(u_{\varepsilon_k}, Y_\varepsilon + x_2) \\ &\le F_0(X_{x_1,\xi,y,z}, Y_\varepsilon + x_2) \ . \end{aligned}$$

So,

$$\frac{1}{|Y_\varepsilon|}\int_{Y_\varepsilon + x_0 + x_2} f_{x_0 + x_1}(x,\xi,y,z)dx \le \frac{1}{|Y_\varepsilon|}\int_{Y_\varepsilon + x_2} f_{x_1}(x,\xi,y,z)dx \ .$$

Making ε go to zero, we get

$$f_{x_0+x_1}(x_0 + x_2,\xi,y,z) \le f_{x_1}(x_2,\xi,y,z)$$

which implies

$$f(x_0 + x_1, x_0 + x_2, \xi, y, z) = f(x_1, x_2, \xi, y, z)$$

i.e. $x \mapsto f(x,x,\xi,y,z) = f_0(x,\xi,y,z)$ is constant.

Step 3.

$$f_0(z) = \lim_{\varepsilon \to 0} \operatorname*{Min}_{u \in W_Y} \frac{1}{|Y|} \int_Y f(\frac{x}{\varepsilon}, D^2u + z)dx \ .$$

(W_Y is the space of Y-periodic functions.)

Clearly since f_0 is convex

$$f_0(z) = \operatorname*{Min}_{u \in W_Y} \frac{1}{|Y|} \int_Y f_0(x,D^2u + z)dx$$

$$= \operatorname*{Min}_{u \in K_z} \frac{1}{|Y|} \int_Y f_0(x,D^2u)dx = \frac{1}{|Y|} \operatorname*{Min}_{u \in K_z} F_0(u,Y)$$

where K_z is the closed convex set

$$K_z = \{u + \frac{1}{2} \sum z_{ij}x_ix_j / u \in W_Y\} = \{v / v - \frac{1}{2} \sum z_{ij}x_ix_j \in W_Y\} \ .$$

Since

$$F_0(u,\Omega) = \Gamma^-(w - W_0^{2,p}(\Omega))F_{\varepsilon_k}(u,\Omega)$$

it follows that

$$(F_0 + \delta_{K_z})(u,Y) = \Gamma^-(w - W^{2,p}(Y))(F_{\varepsilon_k} + \delta_{K_z})(u,Y)$$

and since the functionals are uniformly coercive on $W^{2,p}(Y)$

$$f_0(z) = \lim_{\varepsilon_k \to 0} \operatorname*{Min}_{u \in W_Y} \frac{1}{|Y|} \int_Y f\Big(\frac{x}{\varepsilon_k}, D^2u + z\Big)dx \ .$$

Step 4.

$$\lim_{\varepsilon \to 0} \operatorname*{Min}_{u \in W_Y} \frac{1}{|Y|} \int_Y f(\frac{x}{\varepsilon}, D^2u + z)dx = \operatorname*{Min}_{u \in W_Y} \frac{1}{|Y|} \int_Y f(x,D^2u + z)dx \ .$$

This follows from the equality:

$$\forall h \in \mathbb{N} \quad \operatorname*{Min}_{u \in W_Y} \frac{1}{|Y|} \int_Y f(hx,D^2u + z)dx = \operatorname*{Min}_{u \in W_Y} \frac{1}{|Y|} \int_Y f(x,D^2u + z)dx \ .$$

Let us call

$$M_h = \operatorname*{Min}_{u \in W_Y} \frac{1}{|Y|} \int_Y f(hx, D^2u(x) + z)dx$$

and let u_h be a minimizing point; let us prove first that:

a) $M_1 \geq M_h$.

Let us define

$$\tilde{u}(x) = \frac{1}{h^2} u_1(hx)$$

and let us extend it to Y by periodicity.

$$\frac{1}{|Y|} \int_Y f(hx, D^2\tilde{u} + z)dx = \frac{1}{|Y|} \int_Y f(hx, D^2u_1(hx) + z)dx$$

$$= \frac{1}{h^n} \frac{1}{|Y|} \int_{hY} f(x, D^2u_1(x) + z)dx$$

and since u_1 is Y-periodic

$$= \frac{1}{|Y|} \int_Y f(x, D^2u_1(x) + z)dx = M_1 .$$

Since $\tilde{u}$ is Y-periodic $M_h \leq M_1$.

b) $M_h \geq M_1$.

Let u_h be a solution of M_h and

$$v_h(x) = \frac{1}{h^n} \sum_{(i_1,\ldots,i_n)=0}^{h-1} u_h\left(x_1 + \frac{i_1}{h}, \ldots, x_n + \frac{i_n}{h}\right) .$$

Clearly v_h is $\frac{1}{h}$ Y-periodic, so $\tilde{u}(x) = h^2 v_h(\frac{x}{h})$ is Y-periodic.

$$\frac{1}{|Y|} \int_Y f(x, D^2\tilde{u}(x) + z)dx = \frac{1}{|Y|} \int_Y f(x, D^2v_h(\tfrac{x}{h}) + z)dx$$

$$= \frac{1}{|Y|} h^n \int_{\frac{1}{h}Y} f(hx, D^2v_h(x) + z)dx$$

and since v_h is $\frac{1}{h}$ Y-periodic

$$= \frac{1}{|Y|} \int_Y f(hx, D^2v_h(x) + z)dx ,$$

by convexity

$$\leq \frac{1}{h^n} \sum_{i_1 \ldots i_n=0}^{h-1} \frac{1}{|Y|} \int_Y f\left[hx, D^2u_h\left(x_1 + \frac{i_1}{h}, x_2 + \frac{i_2}{h}, \ldots, x_n + \frac{i_n}{h}\right) + z\right]dx$$

$$\leq \frac{1}{h^n} \sum_{i_1=\dots=i_n=0}^{h-1} \frac{1}{|Y|} \int_{Y+\frac{i}{h}} f(hx - i, D^2 u_h(x) + z)\,dx$$

$$\leq \frac{1}{|Y|} \int_Y f(hx, D^2 u_h(x) + z)\,dx = M_h \ .$$

Therefore $M_1 \leq M_h$ and the conclusion follows.

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We show that many, a priori distinct, problems of homogenization including the case of rapidly oscillating potentials (cf. Benssoussan, J. L. Lions and Papanicolau [2]), can be studied, and the limit problem computed, in a unified way, through general compactness and convergence results for sequences of functionals of calculus of variations. The convergence notion is taken in variational sense, more precisely we use the notion of Γ-convergence introduced by De-Giorgi [4].

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